MHD: Waves and Instabilities

STFC Introductory Summer School
Solar and Solar-Terrestrial Physics

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21/8/23

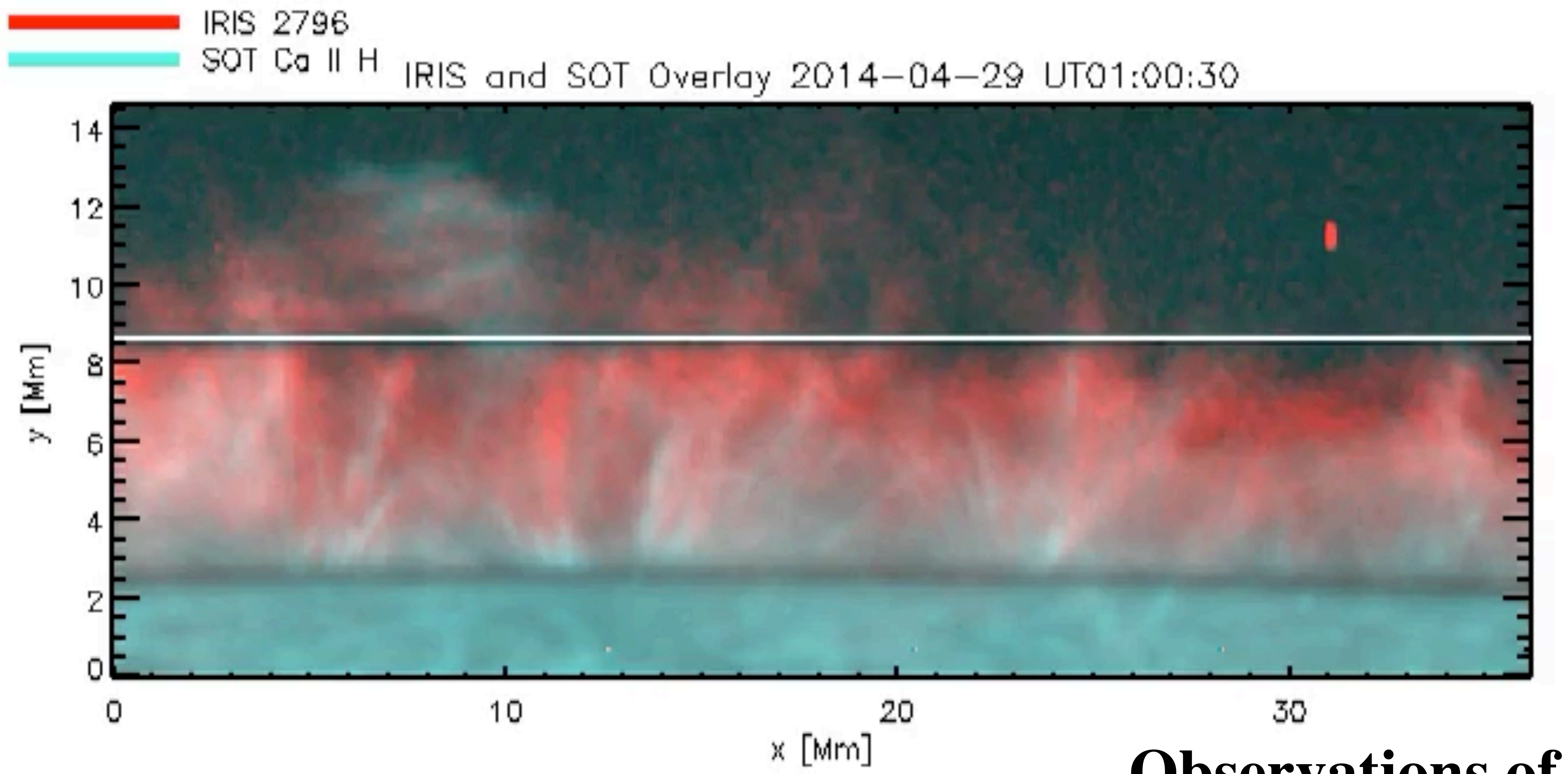
Overview

1. **What are MHD waves and instabilities?**

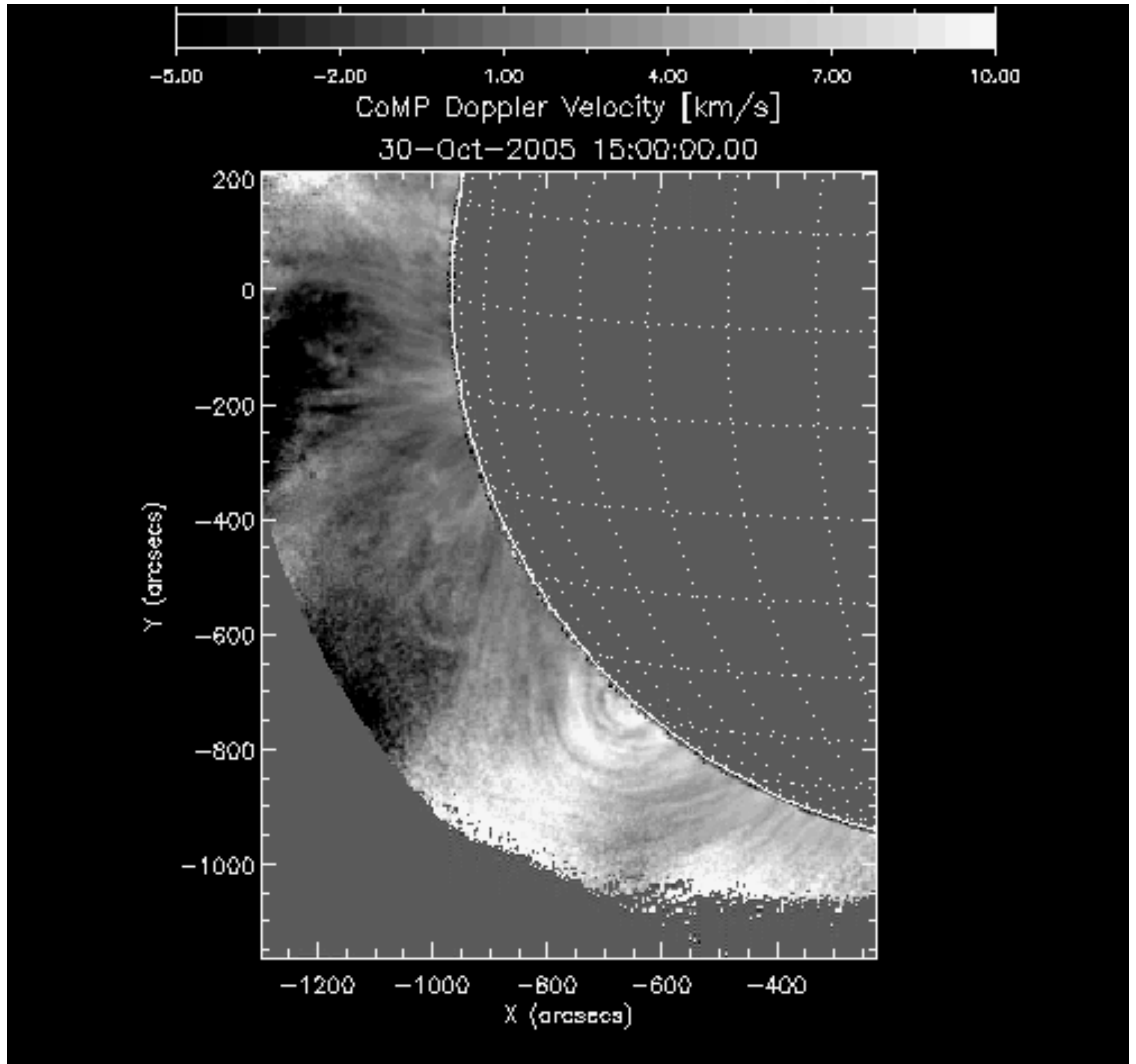
2. **MHD Waves**
   Linearisation, Dispersion Relations, Waves in a Uniform Medium, Non-Uniform Media, Resonances, Phase Mixing, Cylinders, Seismology

3. **MHD Instabilities**
   What do we mean by instability?
   A few different types (tube instabilities, thermal instabilities, Rayleigh-Taylor and Kelvin-Helmholtz instabilities)

4. **Concluding Thoughts**



Observations of Swaying Spicules - IRIS & SOT
Antolin et al. 2018



CoMP Waves
Tomczyk et al. 2007

Waves and Instabilities - What are they?

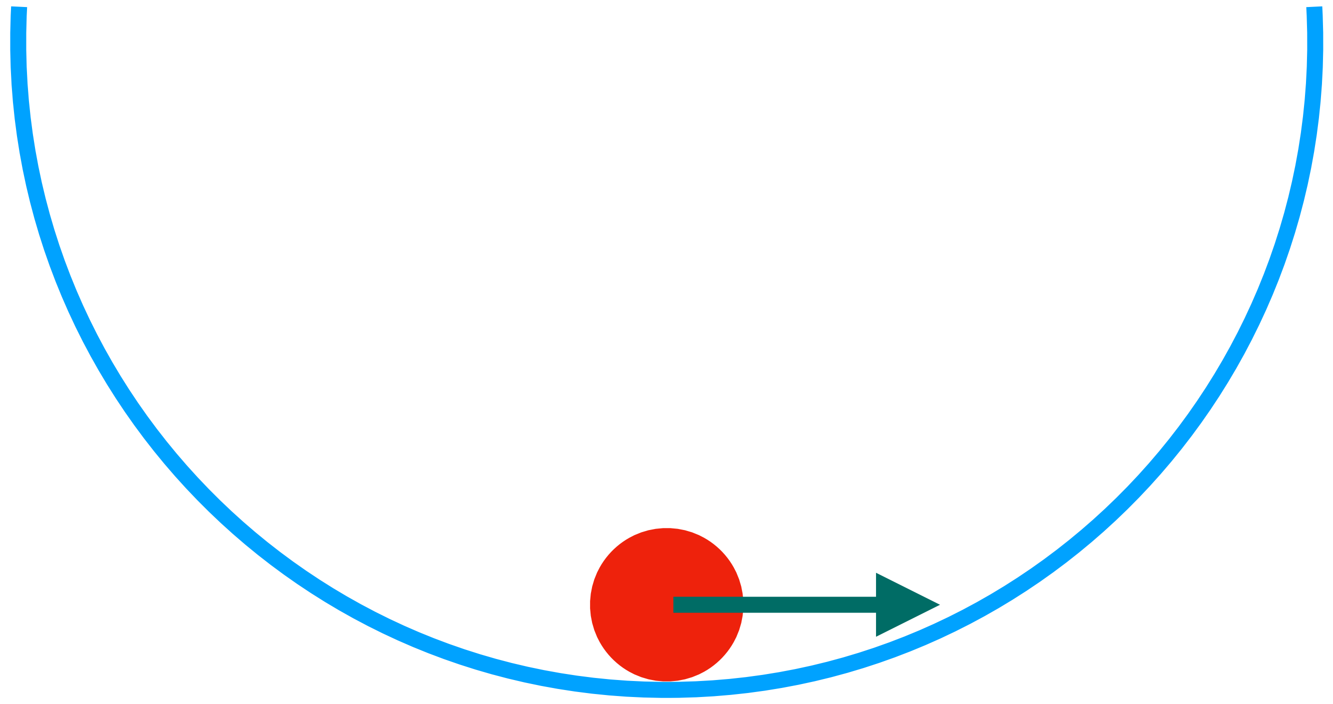
We often begin with an initial equilibrium and ask:
What happens if we give it a push?

Force Balance:

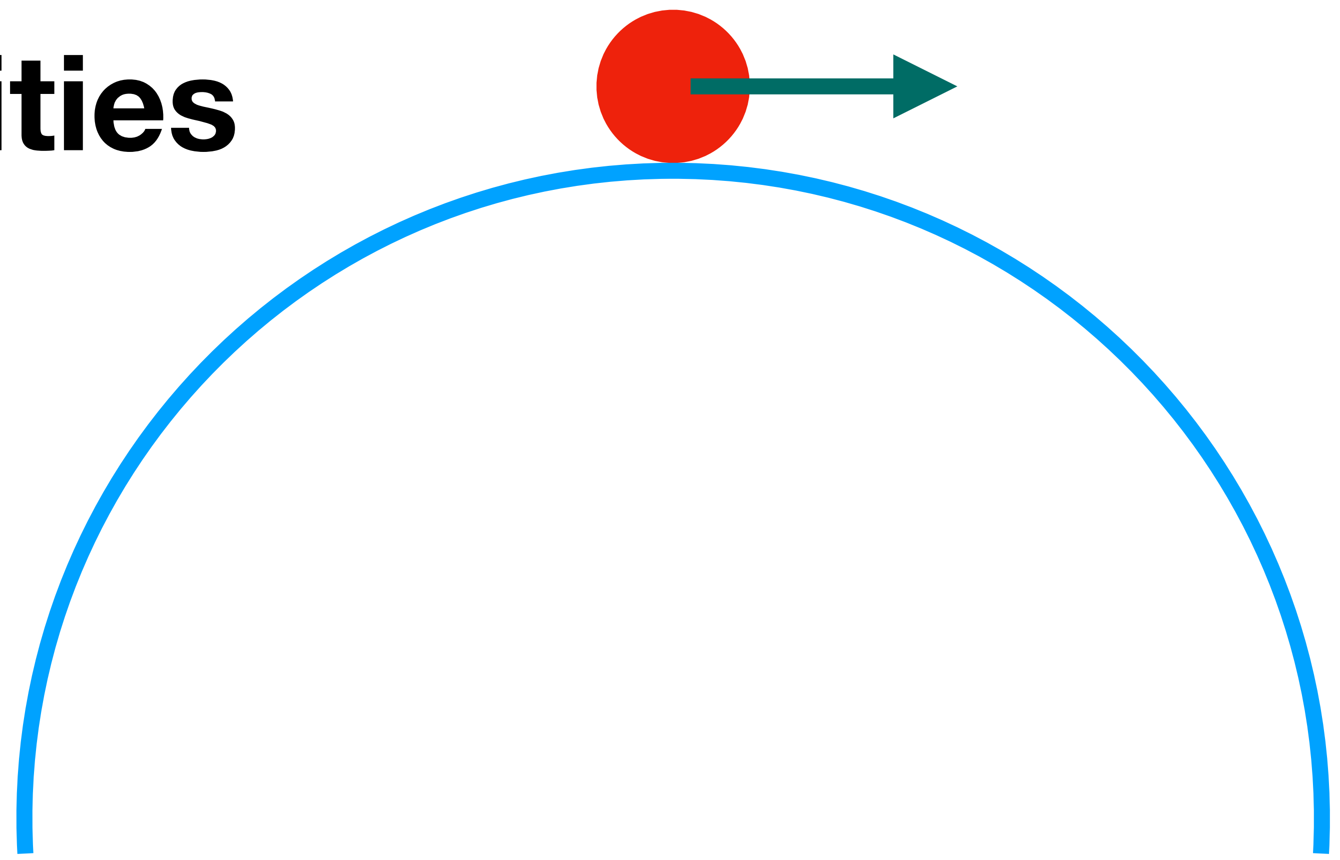
$$0 = \mathbf{j} \times \mathbf{B} - \nabla P + \rho \mathbf{g}$$

If the system is perturbed, the forces will change. Do they act to restore the equilibrium (**waves**) or not (**instability**).

Waves



Instabilities



Waves and Instabilities

How can we describe MHD waves and instabilities mathematically?

1. Initial equilibrium
2. Linearise MHD equations
3. Small perturbation to the system e.g.

$$Ae^{i(\mathbf{k}\cdot\mathbf{r}-\omega t)}$$

4. See what happens!
5. As perturbations can grow exponentially, non-linear analysis may be required

Want to find a relationship between the growth rate, ω, and the wave number, **k**, of the perturbation.

$\boxed{\omega^2 > 0 \qquad \textbf{Waves}}$

Forces oppose any displacement from the equilibrium, creating oscillatory behaviour.

$\omega^2 < 0$ **Instabilities**

Forces enhance any displacement from the equilibrium, creating runaway, exponential growth.

Understanding MHD Waves - Linear Analysis

We can analyse MHD waves and instabilities mathematically by considering small perturbations to an initial equilibrium.

Magnetic Field $\mathbf{B} = \boxed{\mathbf{B_0}} + \boxed{\mathbf{B_1}}$

Velocity $\mathbf{v} = \boxed{\mathbf{v_0}} + \boxed{\mathbf{v_1}}$

Gas Pressure $P = \boxed{P_0} + \boxed{P_1}$

Density $\rho = \boxed{\rho_0} + \boxed{\rho_1}$

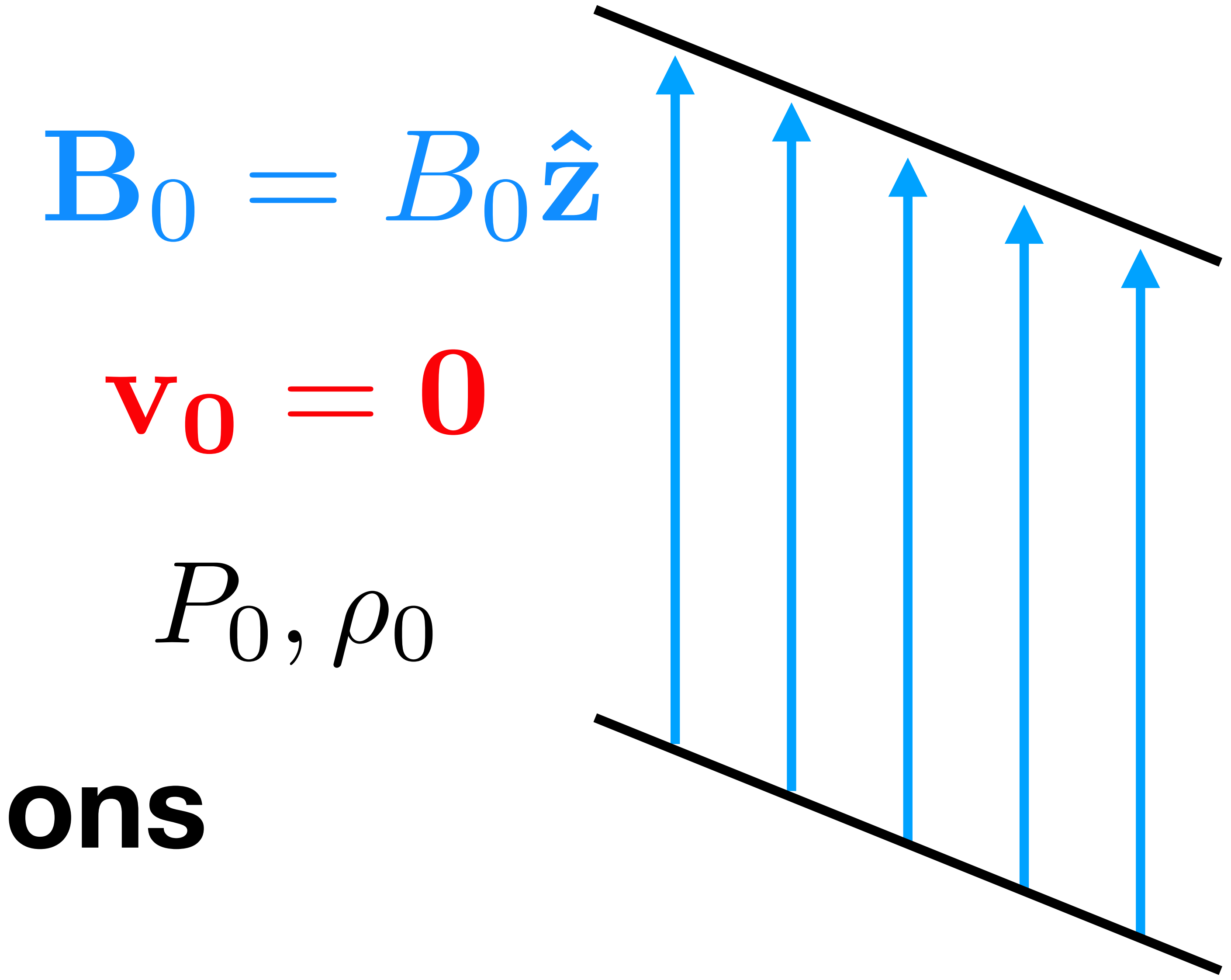
Equilibrium Quantities **Small Perturbations**

We can **linearise** the MHD equations by assuming that **products of perturbed variables are negligible** e.g.

$$\mathbf{v_1} \times \mathbf{B_1} \approx \mathbf{0}$$

Waves in a Uniform Medium

Assume a uniform plasma at rest (**v = 0**) and magnetic field aligned in the *z* direction.



Ideal MHD equations

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{v}) = 0$$

$$\frac{\partial \mathbf{B}}{\partial t} = \nabla \times (\mathbf{v} \times \mathbf{B})$$

$$\rho \frac{\partial \mathbf{v}}{\partial t} = \mathbf{j} \times \mathbf{B} - \nabla P$$

$$\frac{\partial}{\partial t}\left(\frac{P}{\rho^\gamma}\right) + \mathbf{v} \cdot \nabla \left(\frac{P}{\rho^\gamma}\right) = 0$$

$$\nabla \cdot \mathbf{B} = 0$$

Linearised equations

$$\frac{\partial \rho_1}{\partial t} + \rho_0 \nabla \cdot \mathbf{v}_1 = 0$$

$$\frac{\partial \mathbf{B}_1}{\partial t} = \nabla \times (\mathbf{v}_1 \times \mathbf{B}_0)$$

$$\rho_0 \frac{\partial \mathbf{v}_1}{\partial t} = \mathbf{j}_1 \times \mathbf{B}_0 - \nabla P_1$$

$$\frac{\partial}{\partial t}\left\{P_1 - \left(\frac{\gamma P_0}{\rho_0}\right)\rho_1\right\} = 0$$

$$\nabla \cdot \mathbf{B}_1 = 0$$

Waves in a Uniform Medium

Assume Fourier components for the perturbations of the form:

$$Ae^{i(\mathbf{k}\cdot\mathbf{r}-\omega t)}$$

and substitute into linearised equations.

1. Time derivatives: $\frac{\partial}{\partial t}$

Contribute: $-i\omega$

2. Spatial derivatives:

$$\nabla, \nabla\cdot, \nabla\times$$

Contribute:

$$i\mathbf{k}, i\mathbf{k}\cdot, i\mathbf{k}\times$$

$$\frac{\partial \rho_1}{\partial t} + \rho_0 \nabla\cdot\mathbf{v}_1 = 0 \longrightarrow \omega\rho_1 = \rho_0 \mathbf{k}\cdot\mathbf{v}_1$$

$$\frac{\partial \mathbf{B}_1}{\partial t} = \nabla\times(\mathbf{v}_1\times\mathbf{B}_0) \longrightarrow -\omega\mathbf{B}_1 = \mathbf{k}\times(\mathbf{v}_1\times\mathbf{B}_0)$$

$$\rho_0\frac{\partial \mathbf{v}_1}{\partial t} = \mathbf{j}_1\times\mathbf{B}_0 - \nabla P_1 \longrightarrow -\omega\rho_0\mathbf{v}_1 = \frac{(\mathbf{k}\times\mathbf{B}_1)\times\mathbf{B}_0}{\mu_0} - P_1\mathbf{k}$$

$$\frac{\partial}{\partial t}\left\{P_1 - \left(\frac{\gamma P_0}{\rho_0}\right)\rho_1\right\} = 0 \longrightarrow P_1 = \left(\frac{\gamma P_0}{\rho_0}\right)\rho_1$$

$$\nabla\cdot\mathbf{B}_1 = 0 \longrightarrow \mathbf{k}\cdot\mathbf{B}_1 = 0$$

Waves in a Uniform Medium

Assume Fourier components for the perturbations of the form:

and subs

Then, algebra and vector identities……….

1. Time de

Contribut

2. Spatial

Contribute:

$$i\mathbf{k}, i\mathbf{k}\cdot, i\mathbf{k}\times$$

$$\frac{\partial}{\partial t}\left\{P_1 - \left(\frac{\gamma P_0}{\rho_0}\right)\rho_1\right\} = 0 \longrightarrow P_1 = \left(\frac{\gamma P_0}{\rho_0}\right)\rho_1$$

$$\nabla \cdot \mathbf{B}_1 = 0 \longrightarrow \mathbf{k} \cdot \mathbf{B}_1 = 0$$

Dispersion Relations

This algebra allows us to eliminate the perturbed variables to find:

$$\boxed{\left(\omega^2 - k^2 c_A^2 \cos^2\theta\right)}\boxed{\left(\omega^4 - \omega^2 k^2 \left(c_s^2 + c_A^2\right) + c_s^2 c_A^2 k^4 \cos^2\theta\right)} = 0$$

Alfvén Speed $c_A^2 = \frac{B_0^2}{\mu_0 \rho_0}$ **Sound Speed** $c_s^2 = \frac{\gamma P_0}{\rho_0}$ **Anisotropy due to** θ

The two brackets provide dispersion relations for Alfvén and magnetoacoustic waves, respectively.

Alfvén

$$\frac{\omega^2}{k^2} = c_A^2 \cos^2\theta$$

Magnetoacoustic (fast and slow)

$$\frac{\omega^2}{k^2} = \frac{1}{2}\left(c_s^2 + c_A^2\right) \pm \frac{1}{2}\sqrt{\left(c_s^2 + c_A^2\right)^2 - 4c_s^2 c_A^2 \cos^2\theta}$$

MHD wave modes

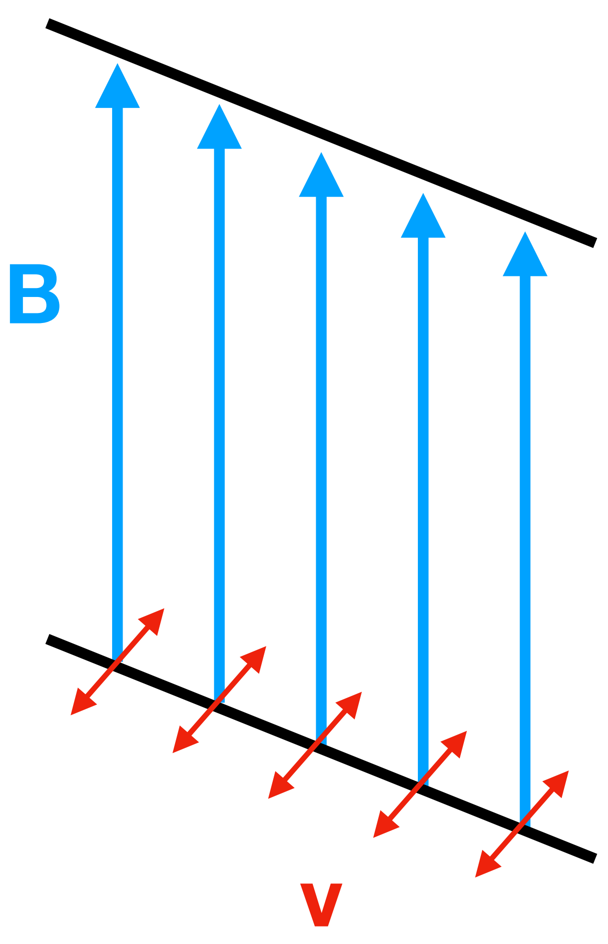
Alfvén

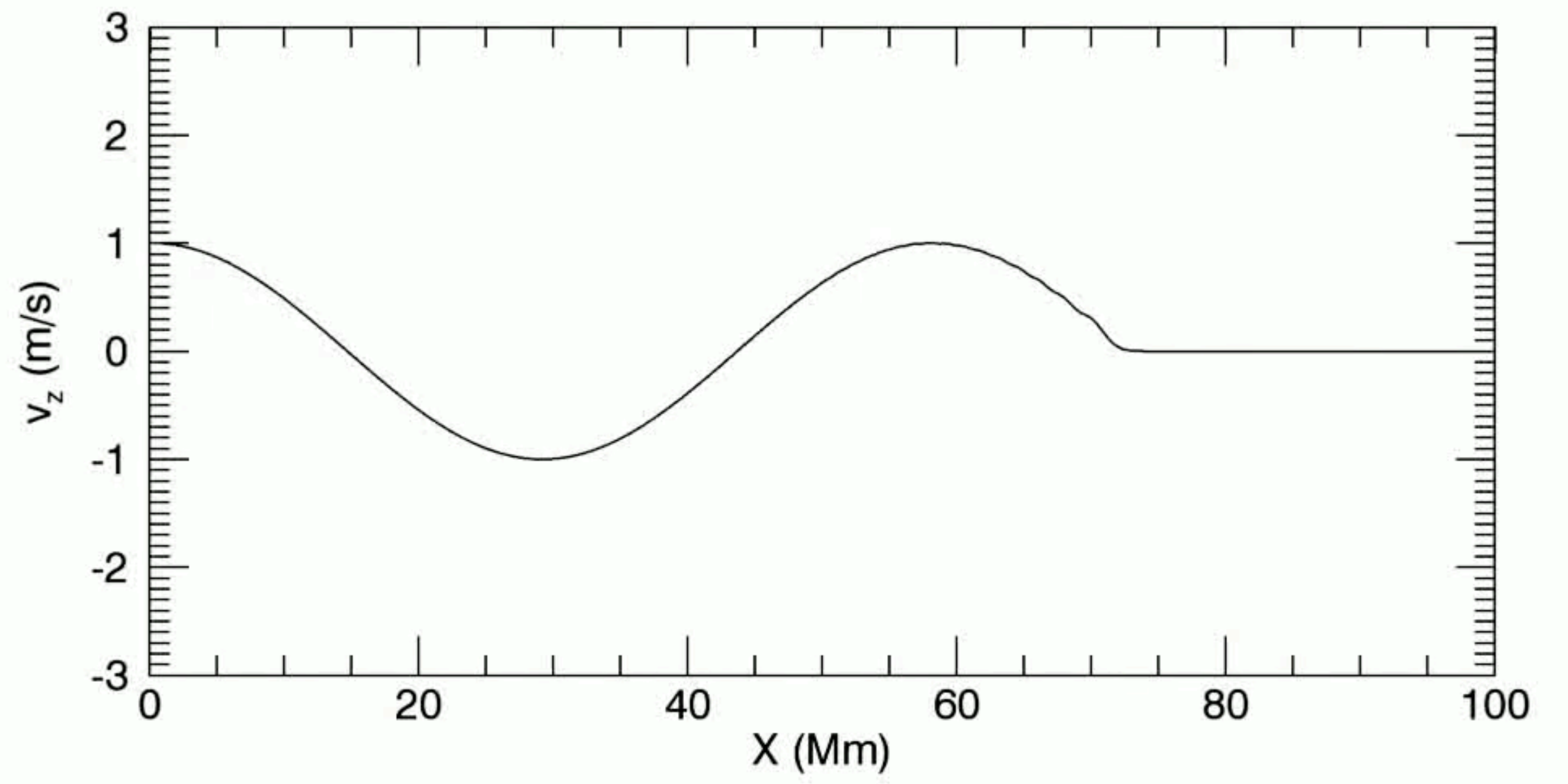
Behave like waves on a string.

Incompressible.

Restoring force is magnetic tension.

Group speed is the Alfvén speed.





Fast and Slow Magnetoacoustic

Compressive waves.

For fast/slow wave, density and field strength are perturbed in/out of phase.

Important Limits

$\theta \to 0$ **Parallel Propagation** Fast and slow waves propagate as pure Alfvén and slow waves.

$\theta \to \pi/2$ **Perpendicular Propagation** Fastest propagation for fast wave. Slow wave phase speed goes to 0.

Cold Limit No gas pressure so sound speed vanishes. No slow wave and fast wave propagates isotropically with the Alfvén speed.

Low β Slow wave confined to propagate along magnetic field lines as an acoustic mode.

MHD wave modes

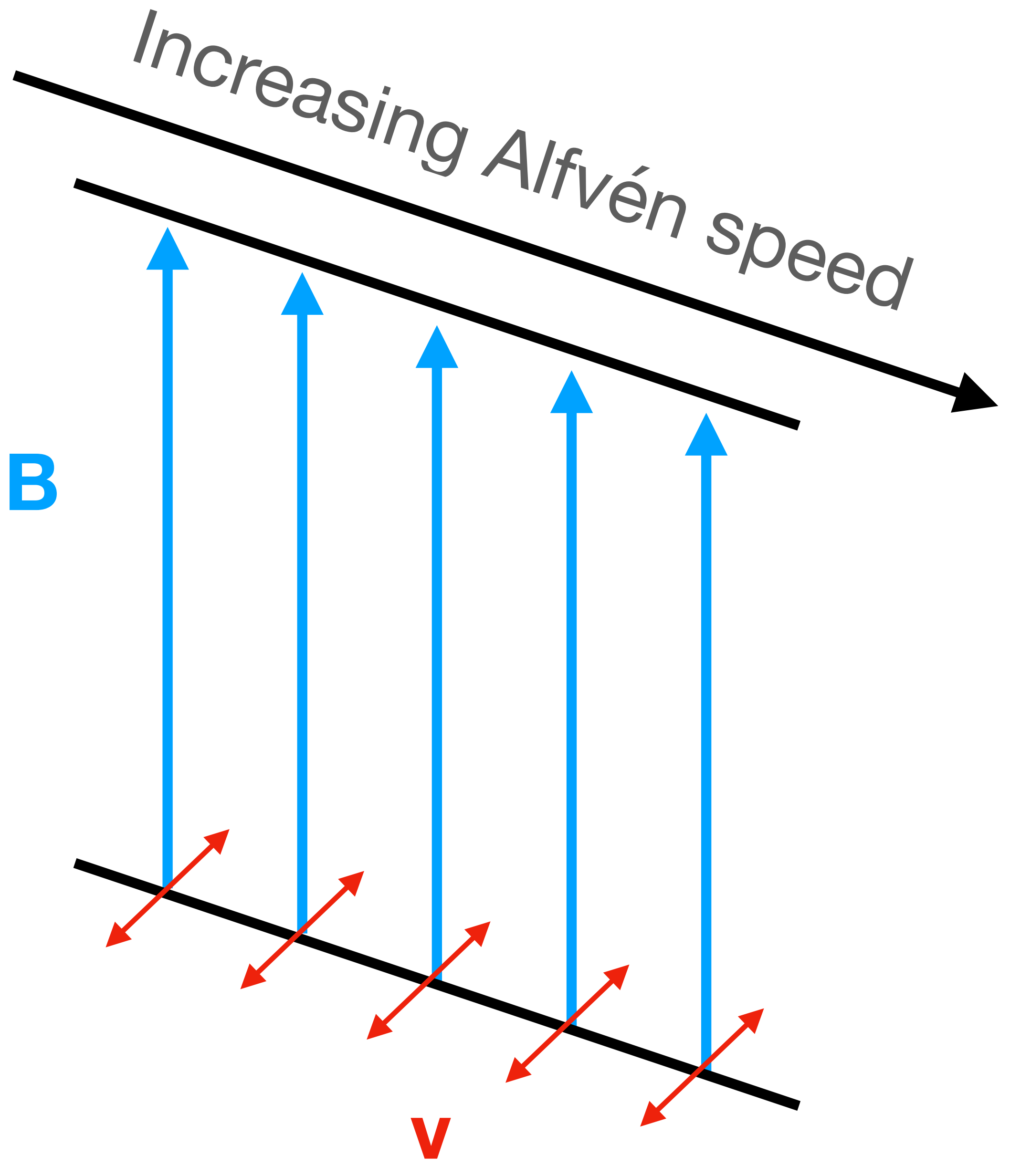
as an acoustic mode.

Thanks to E Enerhaug

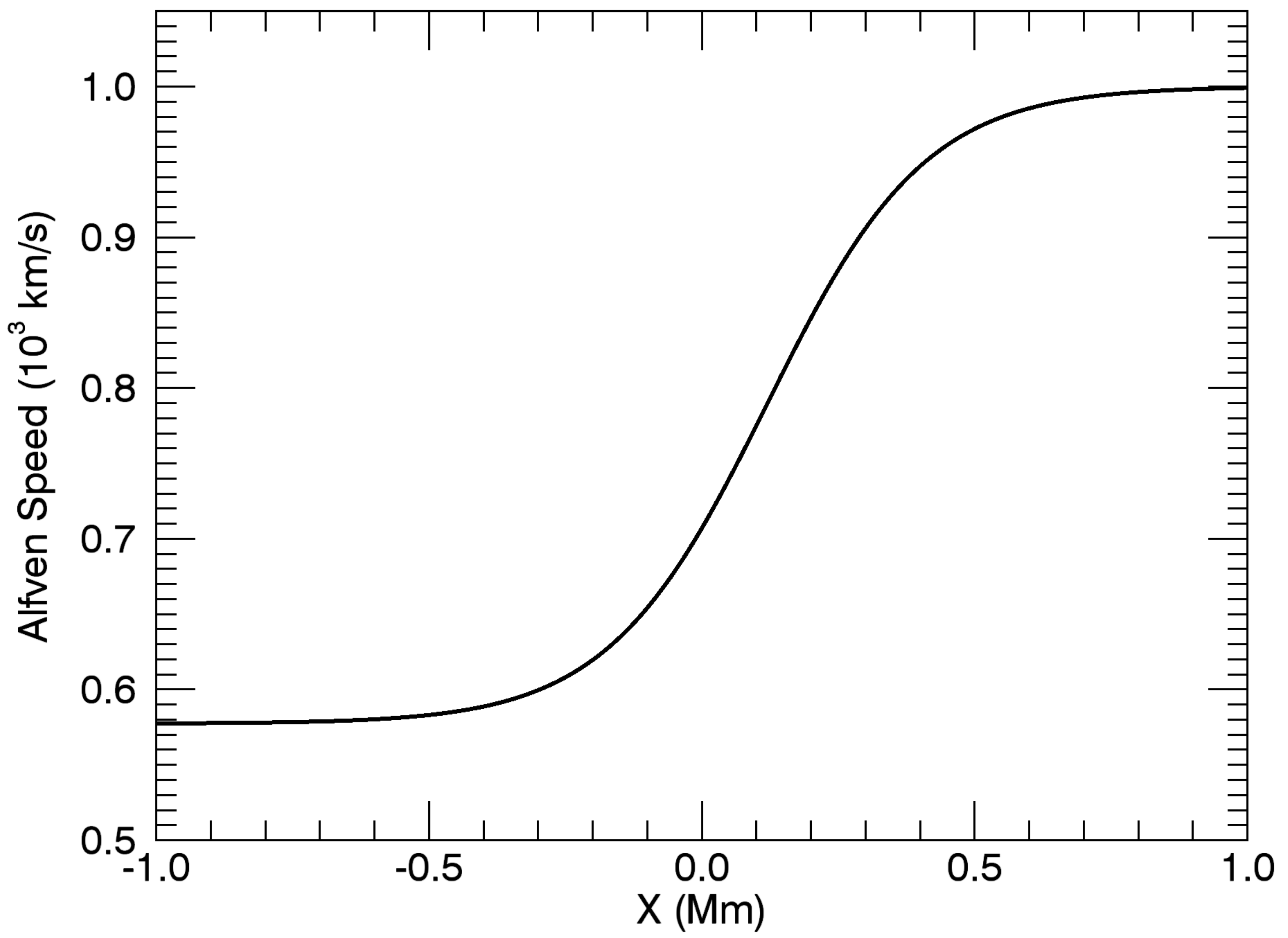
Non-Uniform Medium - Phase Mixing

In fully 3D plasmas, the MHD modes no longer decouple and waves have **mixed properties.** However, if there is an invariant direction, pure Alfvén waves can still exist.

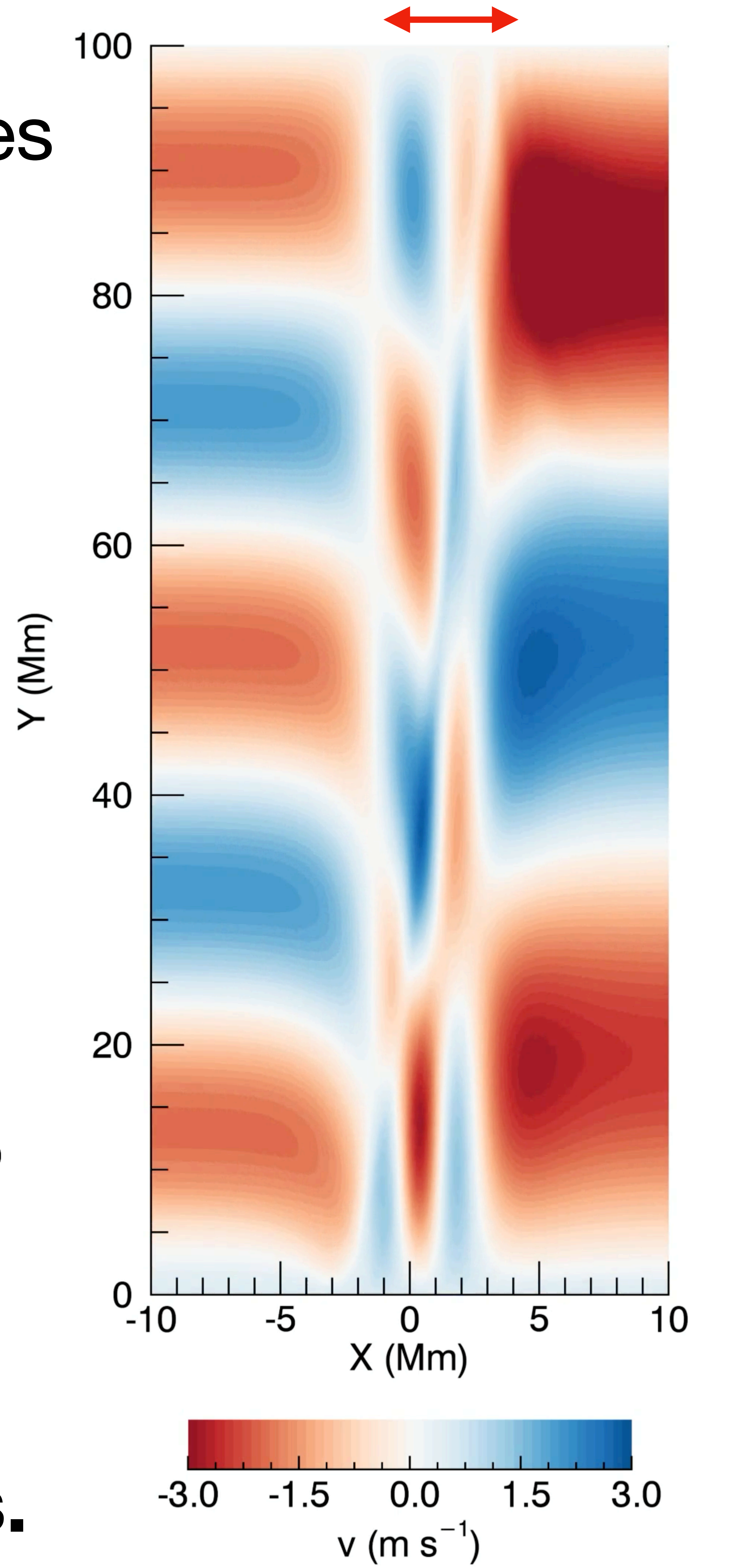
$$v_A = \frac{B}{\sqrt{\mu_0 \rho}}$$



Alfvén Speed Profile



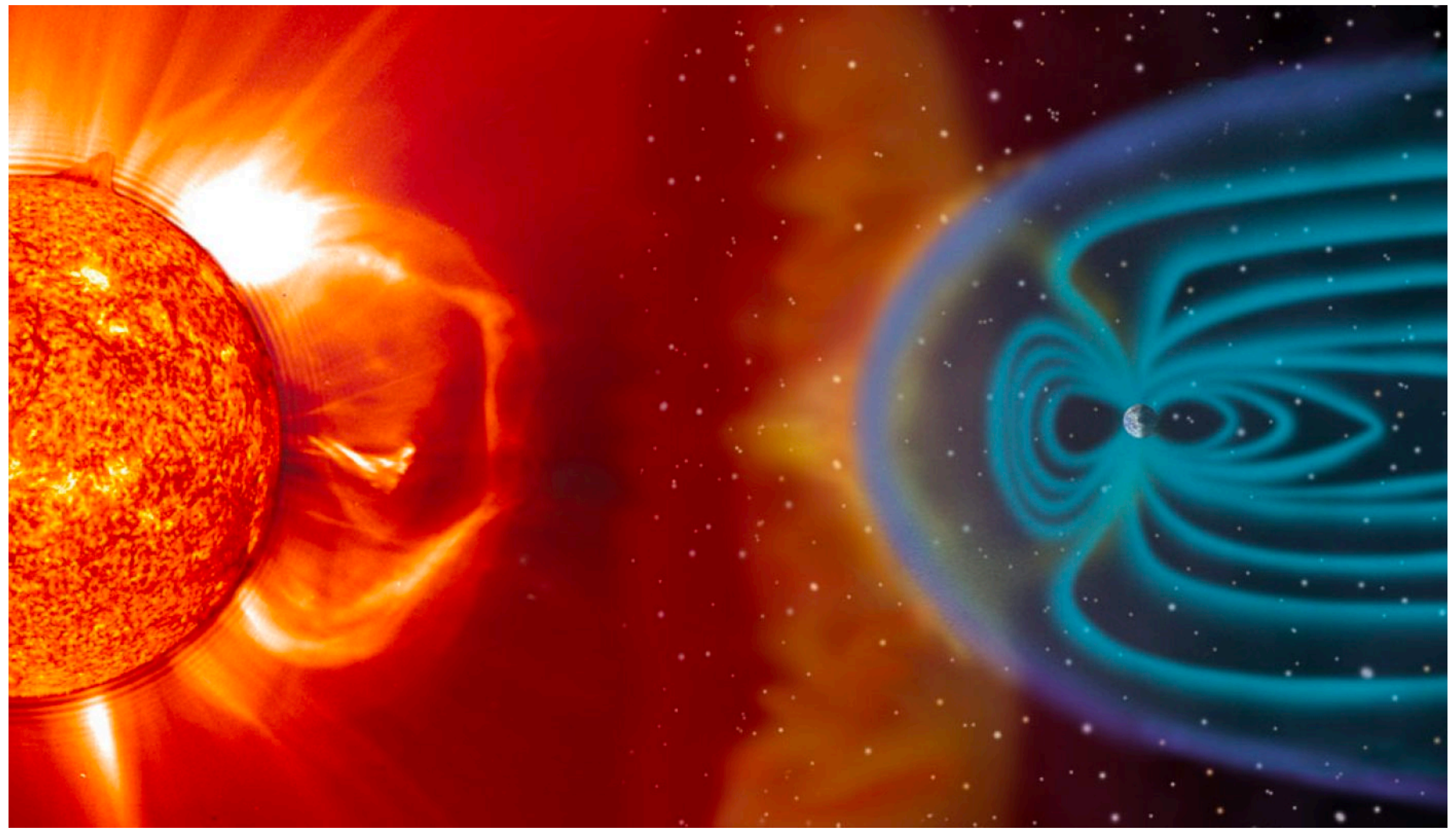
Short transverse length scales
Greater Energy Dissipation



Transverse gradient in the local Alfvén speed. Linear waves confined to individual field lines so out of phase waves on neighbouring field lines produce increasing **small length scales.**

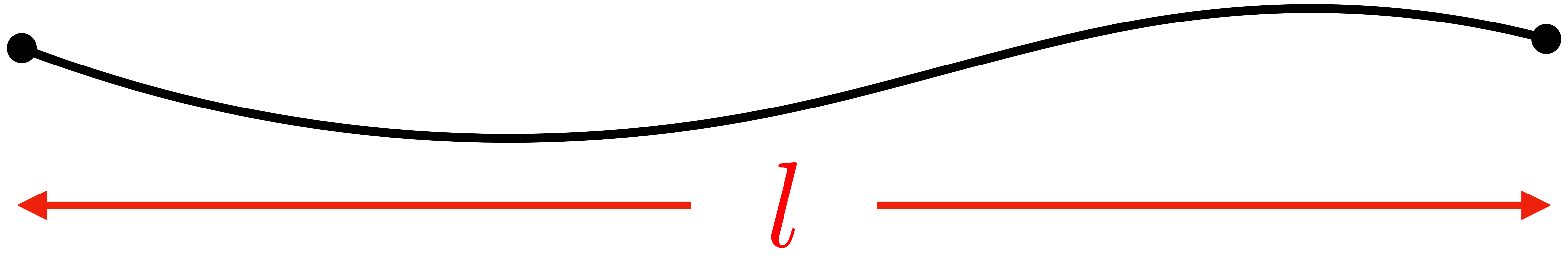
Waves - Resonances

Field lines with fixed ends will have natural frequencies. Oscillatory drivers with power at these frequencies will efficiently excite **resonant standing waves.**



The Sun-Earth System - NASA

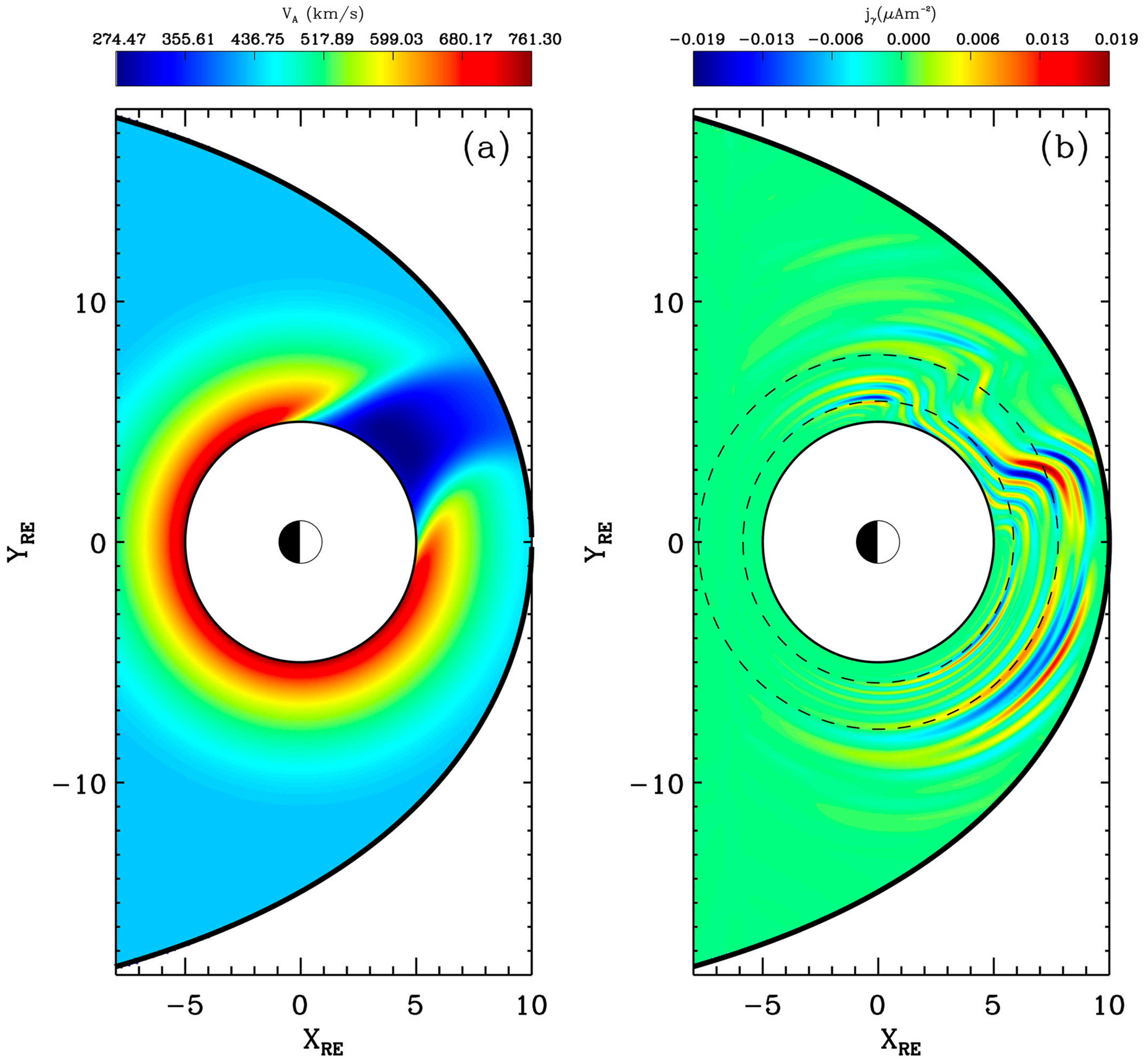
Uniform field line - Alfvén Speed v_A



Period of fundamental mode for uniform string is twice travel time

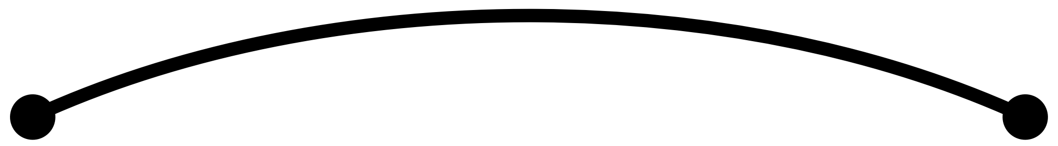
$$\tau = \frac{2l}{v_A}$$

Fundamental frequency is: $\omega = \frac{2\pi}{\tau} = \frac{\pi v_A}{l}$

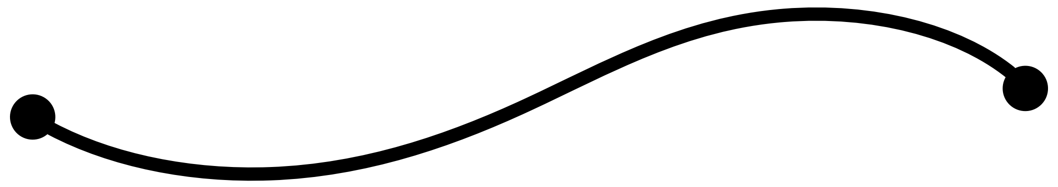


Field line resonances in the Earth's Magnetosphere - Wright & Elsden (2020)

Can also excite **higher harmonics:**

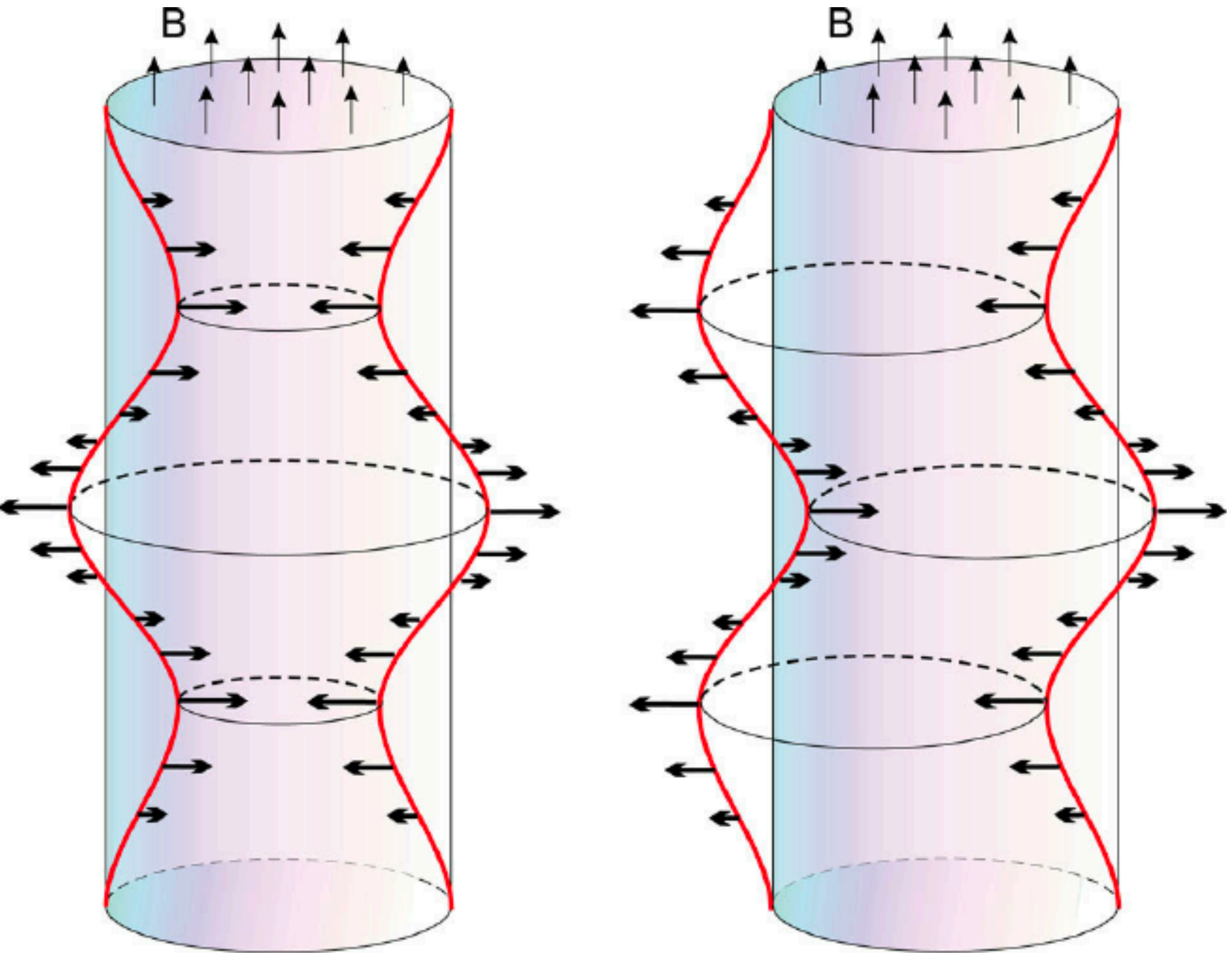


Fundamental



First overtone $\omega = \frac{2\pi v_A}{l}$

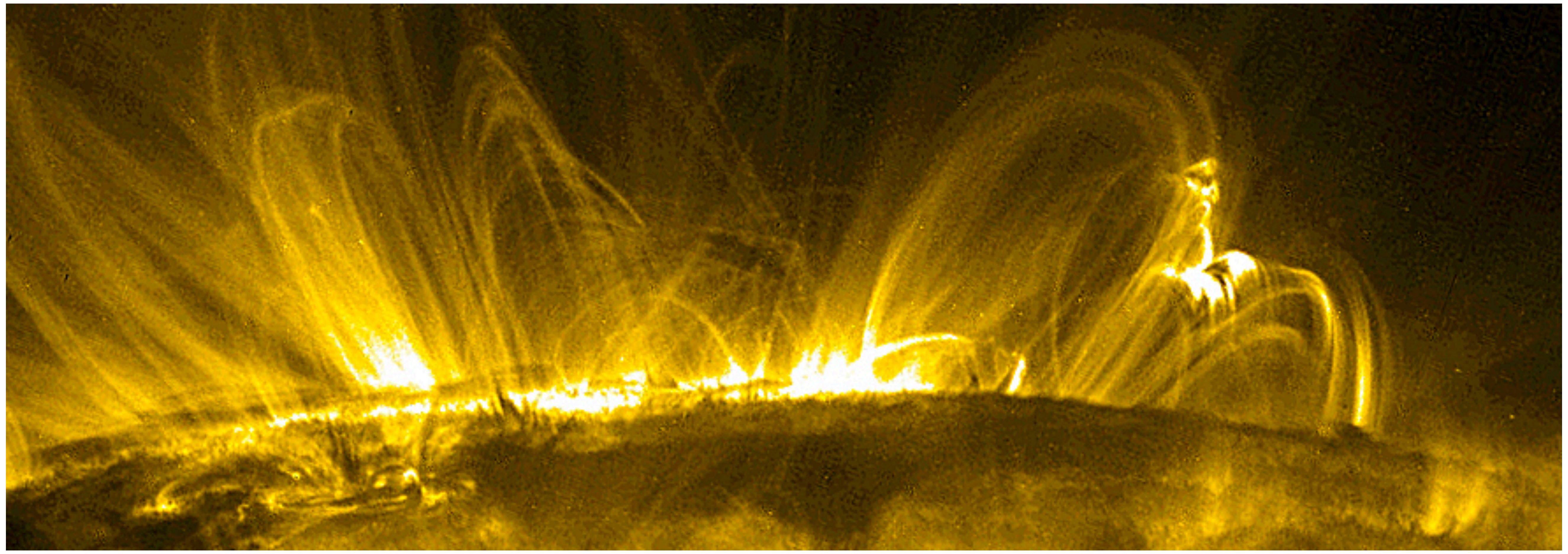
Waves in Cylinders



Sausage Mode

Kink Mode

Morton et al. 2012

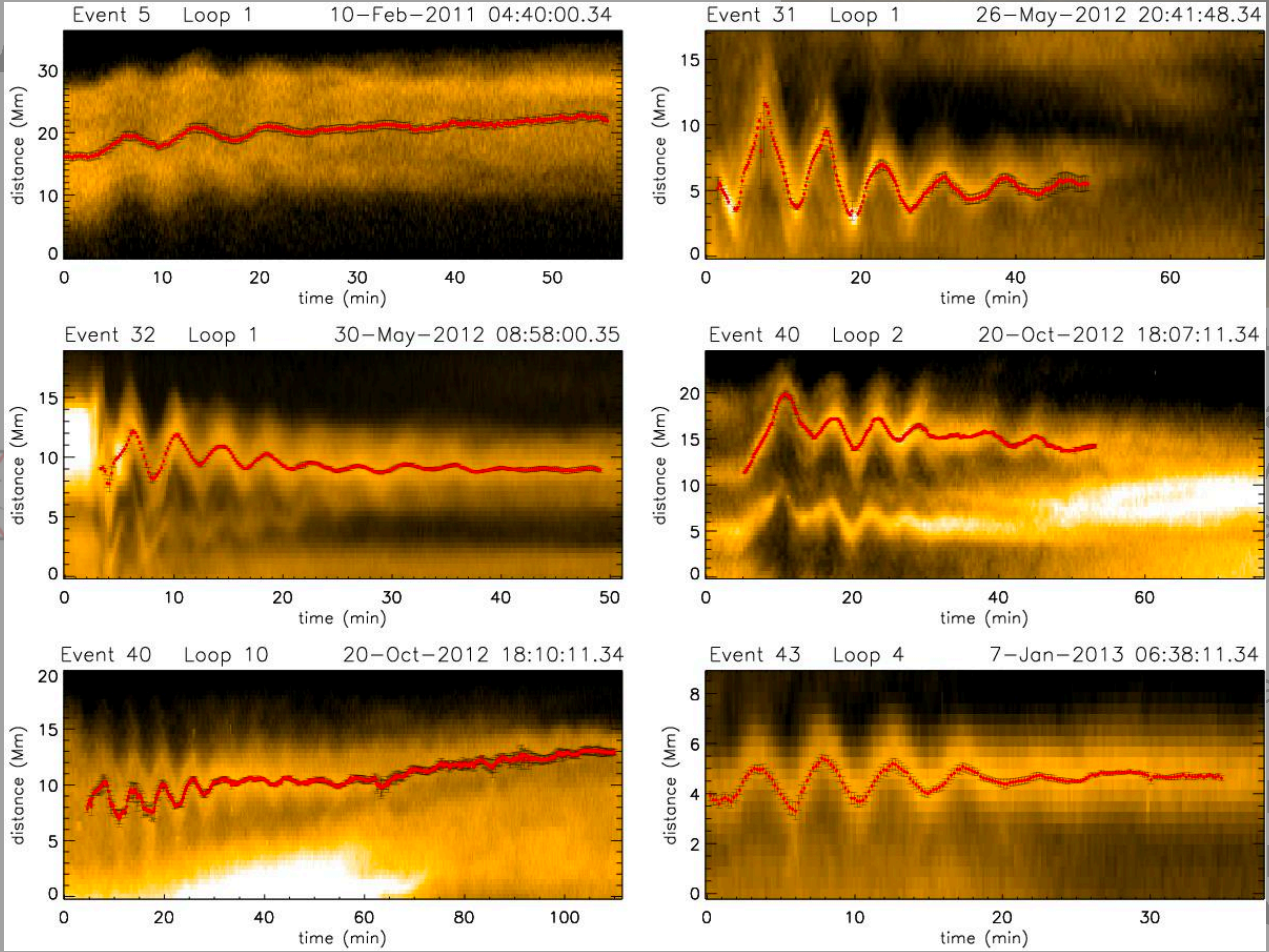


Coronal Loops observed by TRACE

Often model loops as straight cylinders. These support many wave modes, including **sausage**, **kink** and higher order **ballooning/ fluting** modes.

Magnetic tension dominant restoring force.

Weakly compressible kink modes widely observed in Sun's atmosphere. They **damp very rapidly** despite weak dissipation - **why?**



Loops observed by TRACE

aight cylinders.

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er order **ballooning/**

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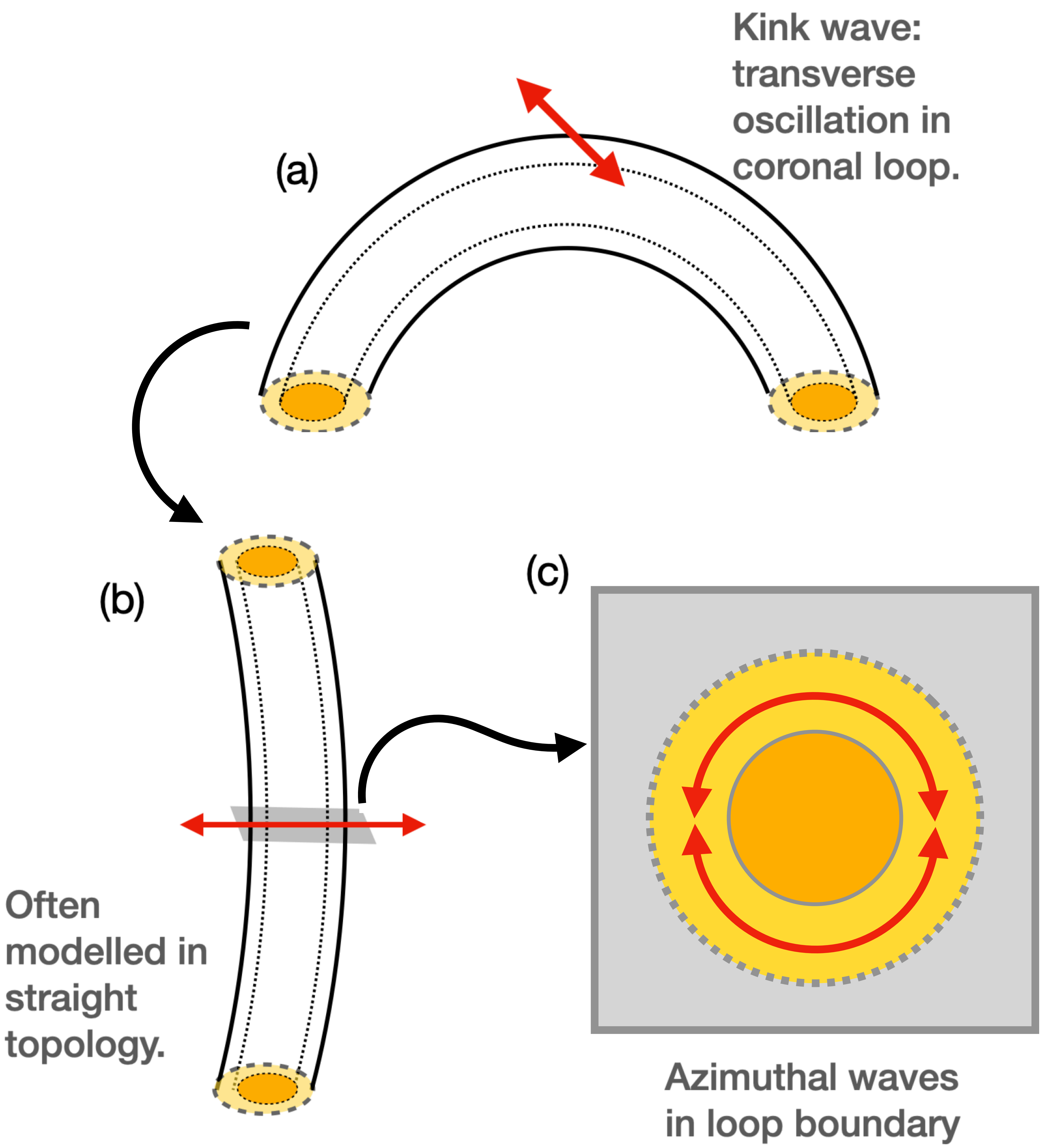
why?

Time-distance maps for kink waves. Pascoe et al. 2015, Goddard et al. 2016

Resonant Absorption



Many observations of oscillating coronal loops. Interpreted as kink modes.

Kink speed:

$$v_k = \sqrt{\frac{\rho_i v_{A_i}^2 + \rho_e v_{A_e}^2}{\rho_i + \rho_e}}$$

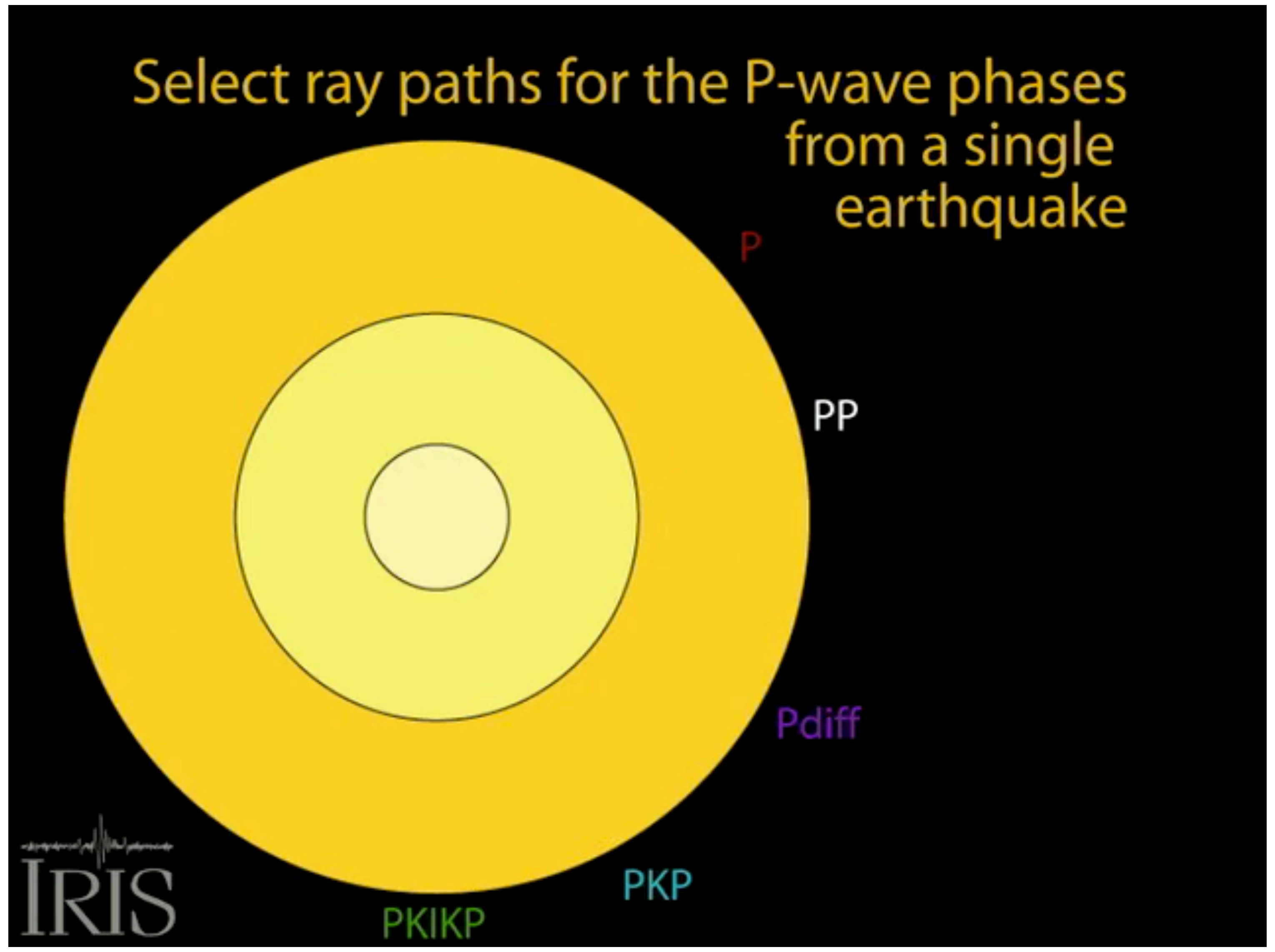
Matches local Alfvén speed in the boundary of loop -> **resonance!**

Energy is efficiently transferred from the global kink mode to localised azimuthal Alfvén waves.

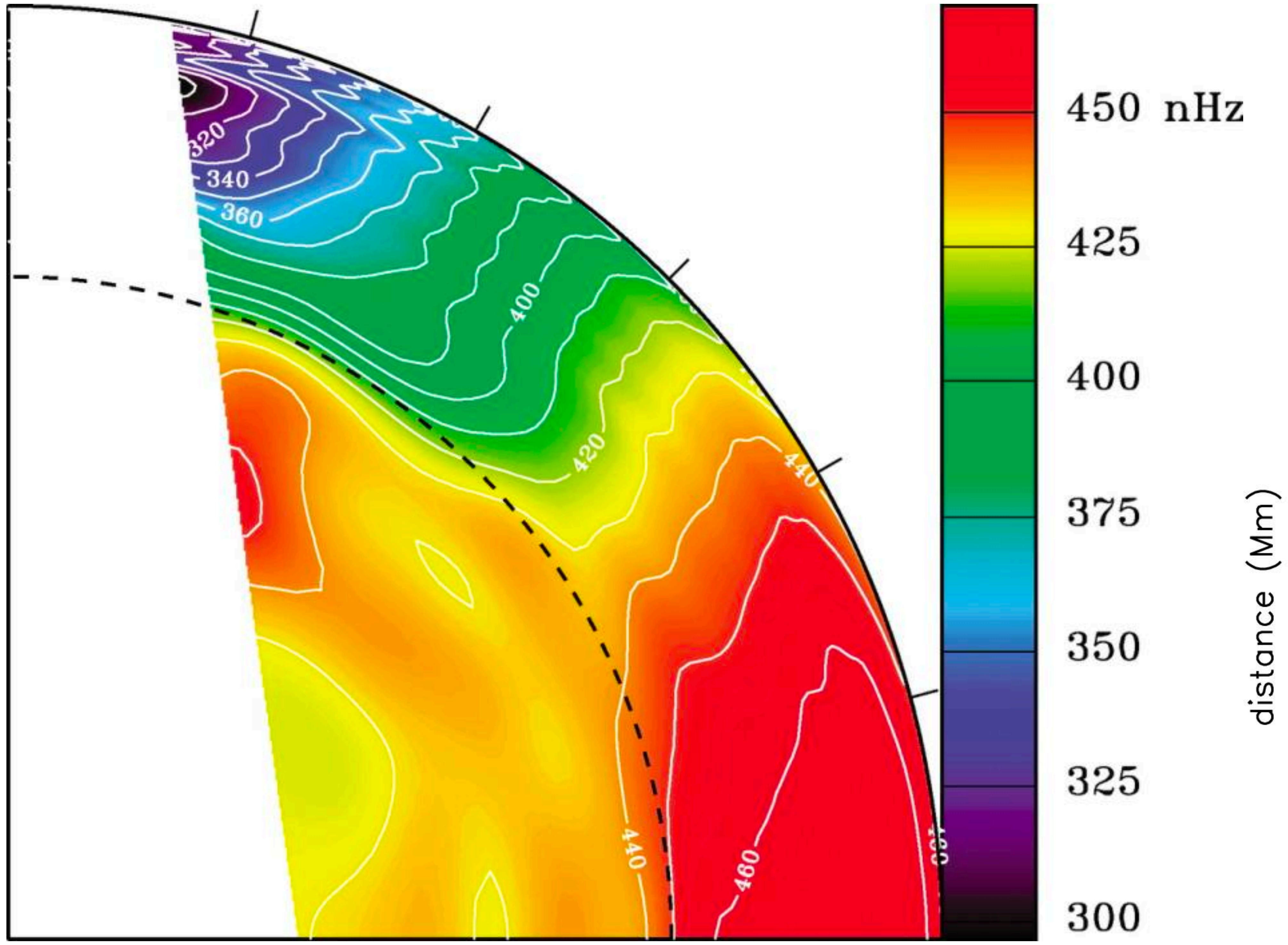
Kink wave damps very quickly but no direct energy dissipation. Can lead to instability as we shall see….!

Waves - Seismology

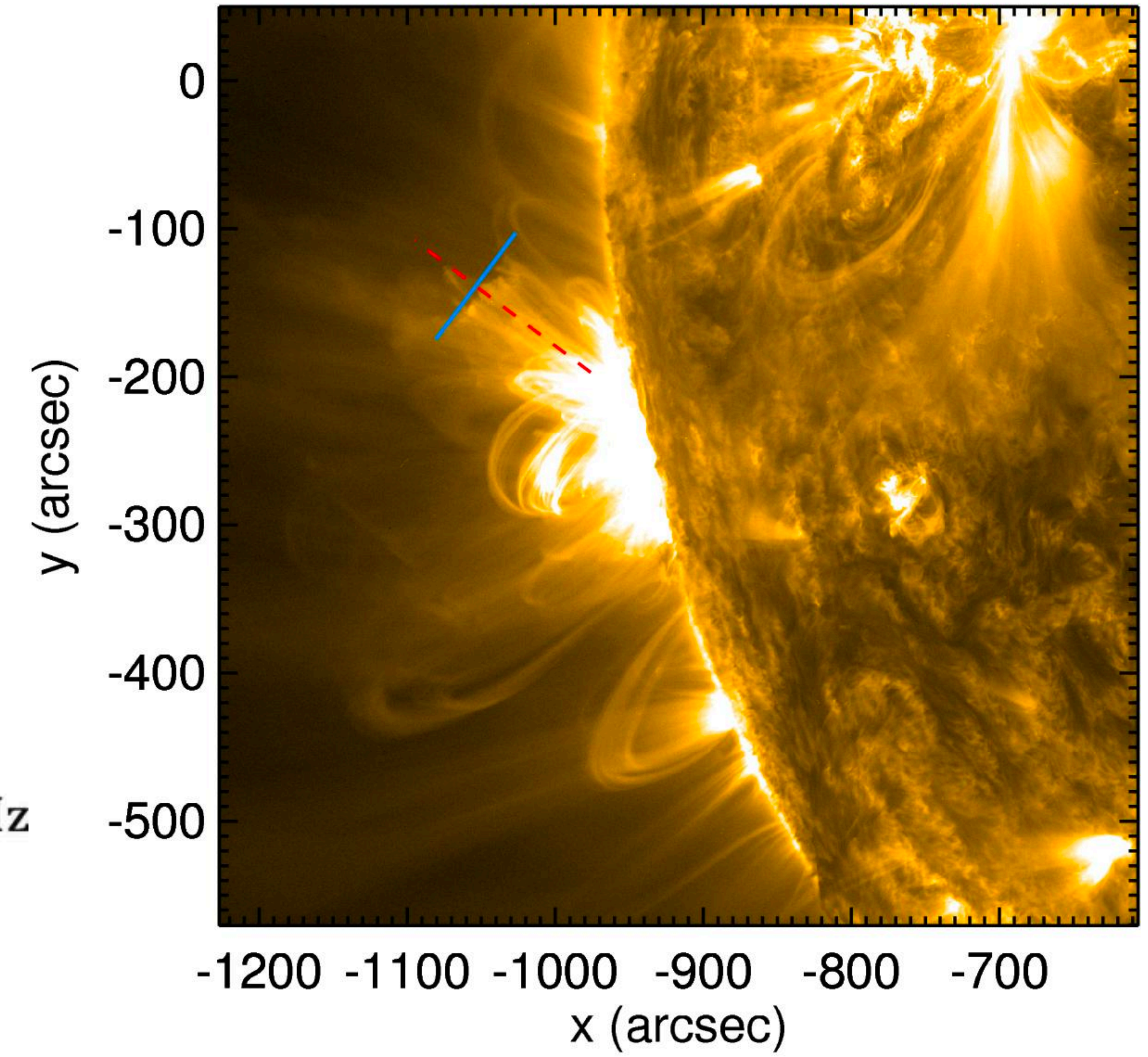
Waves can carry information about their source and the background medium. We can use waves to **infer plasma properties that we cannot measure directly.**

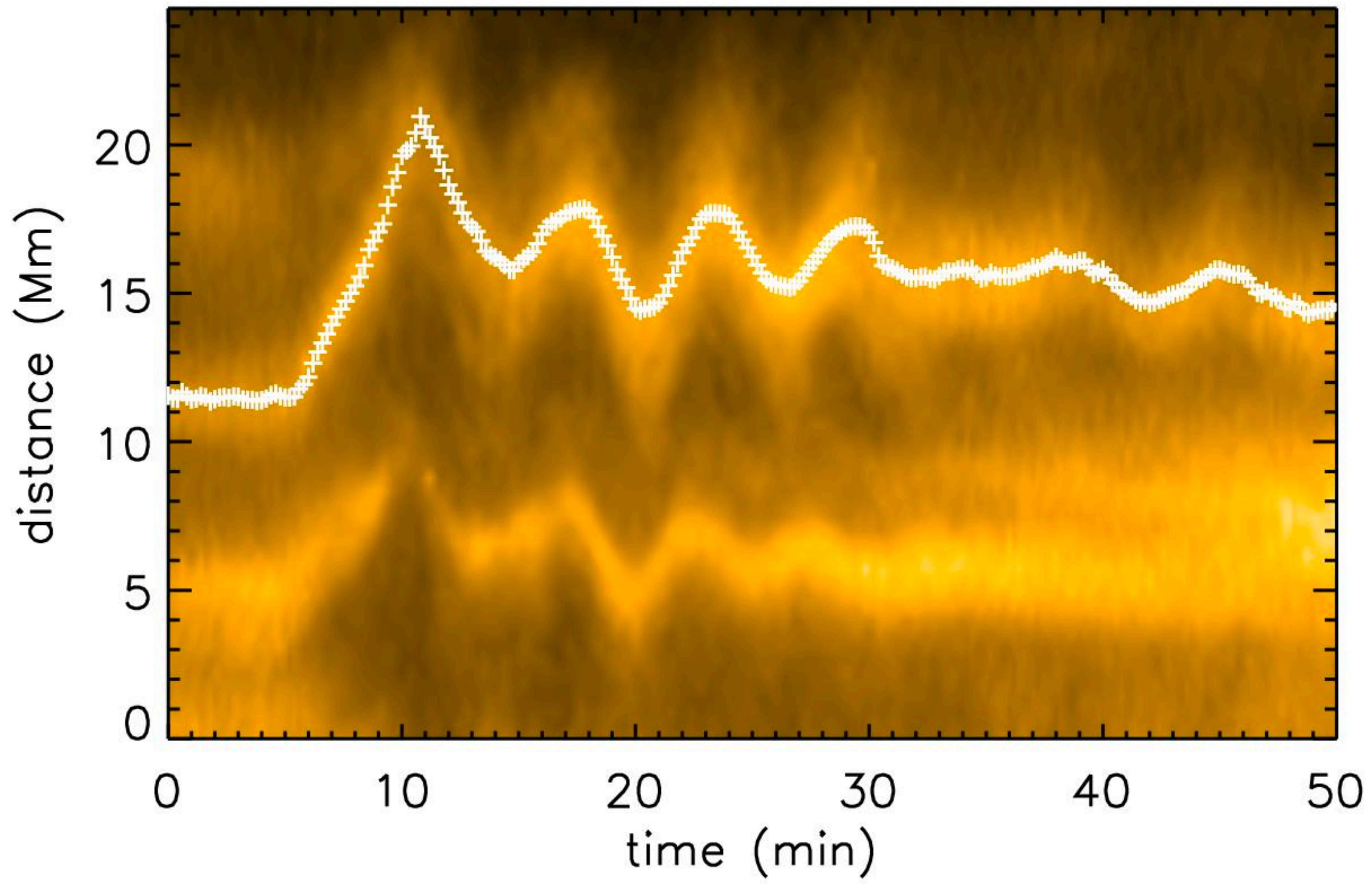


Seismology of the Earth's Interior
IRIS/SAGE



Inferring rotational profile of the solar interior using helioseismology
HAO

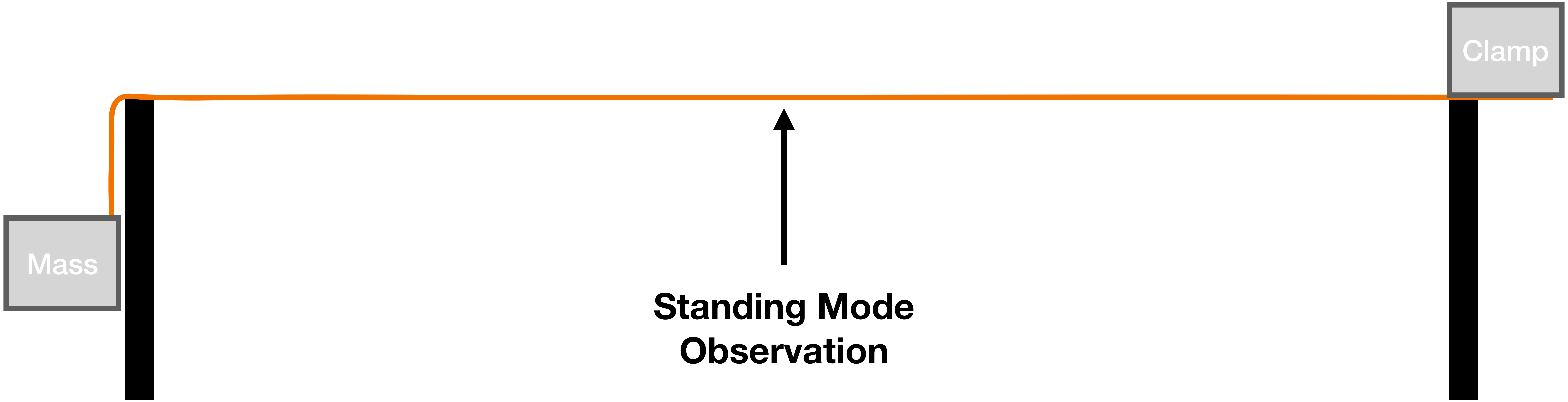




Coronal Seismology with kink oscillations
(Pascoe et al. 2017)

Waves - Seismology in Practice

Can we used waves to estimate the unknown mass?



Wave Speed on Stretched String

$$v = \sqrt{\frac{T}{\mu}}$$

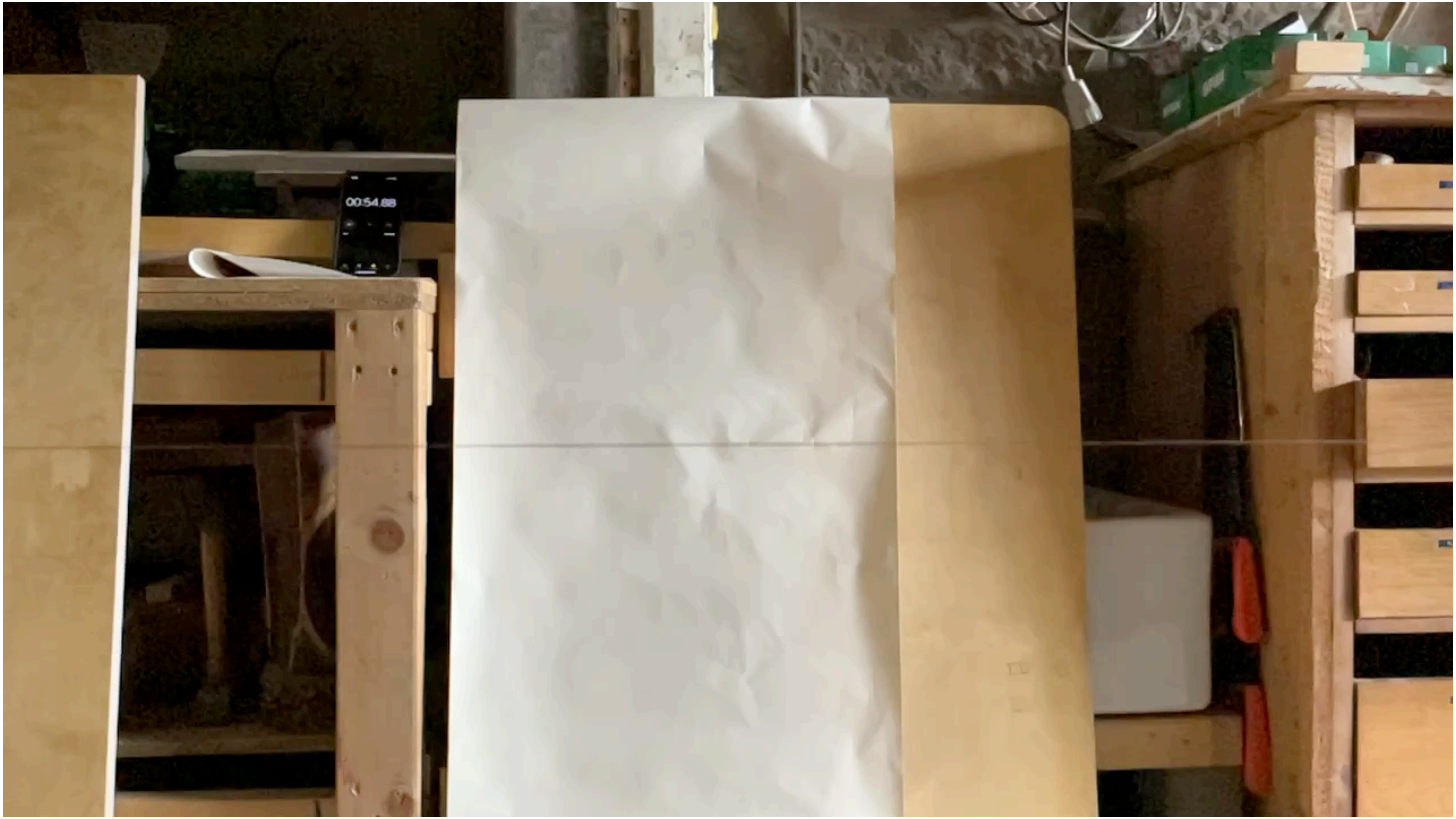
T **Tension** μ **Mass per unit length**

Using Copper Wire:

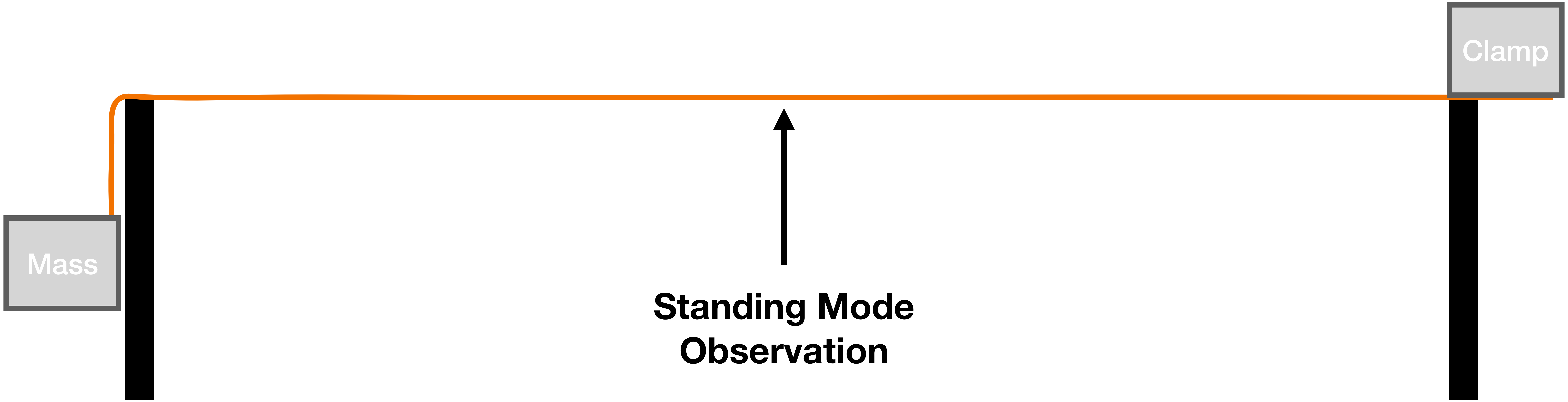
Standard Wire Gauge: 26
Mass per unit length: 0.00146 kg/m

Waves - Seismology in Practice

Standing Mode Observation



For a uniform string, standing mode period is twice the wave travel time along the wire. **What else do you need to know?**



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Instabilities

There are many ways a plasma can be unstable. This list shows a few of them:

In Solar System Science, we may be interested in (e.g.):

Tearing mode instability for reconnection

Convective instability in the solar interior

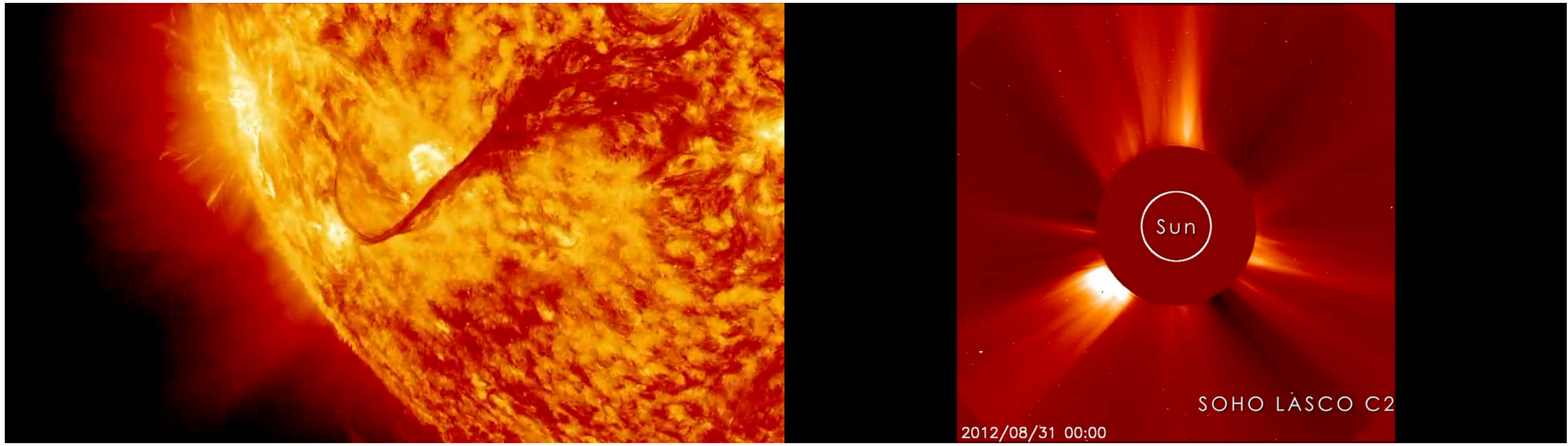
Kelvin-Helmholtz instability on the flanks of magnetospheres

Kink instability as the driver of impulsive events in the corona

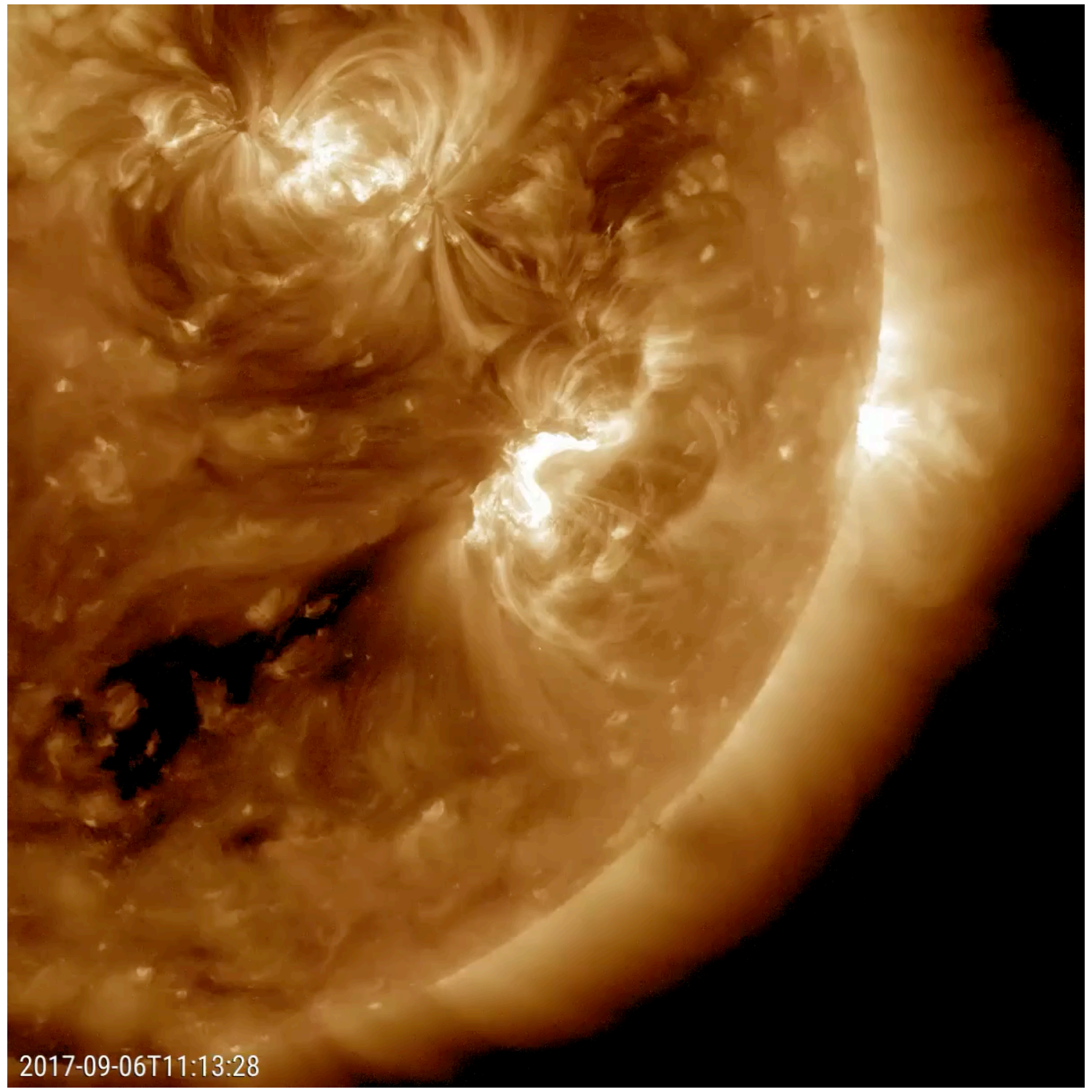
Magnetic buoyancy instability for flux emergence in the Sun

Rayleigh-Taylor instability in prominences

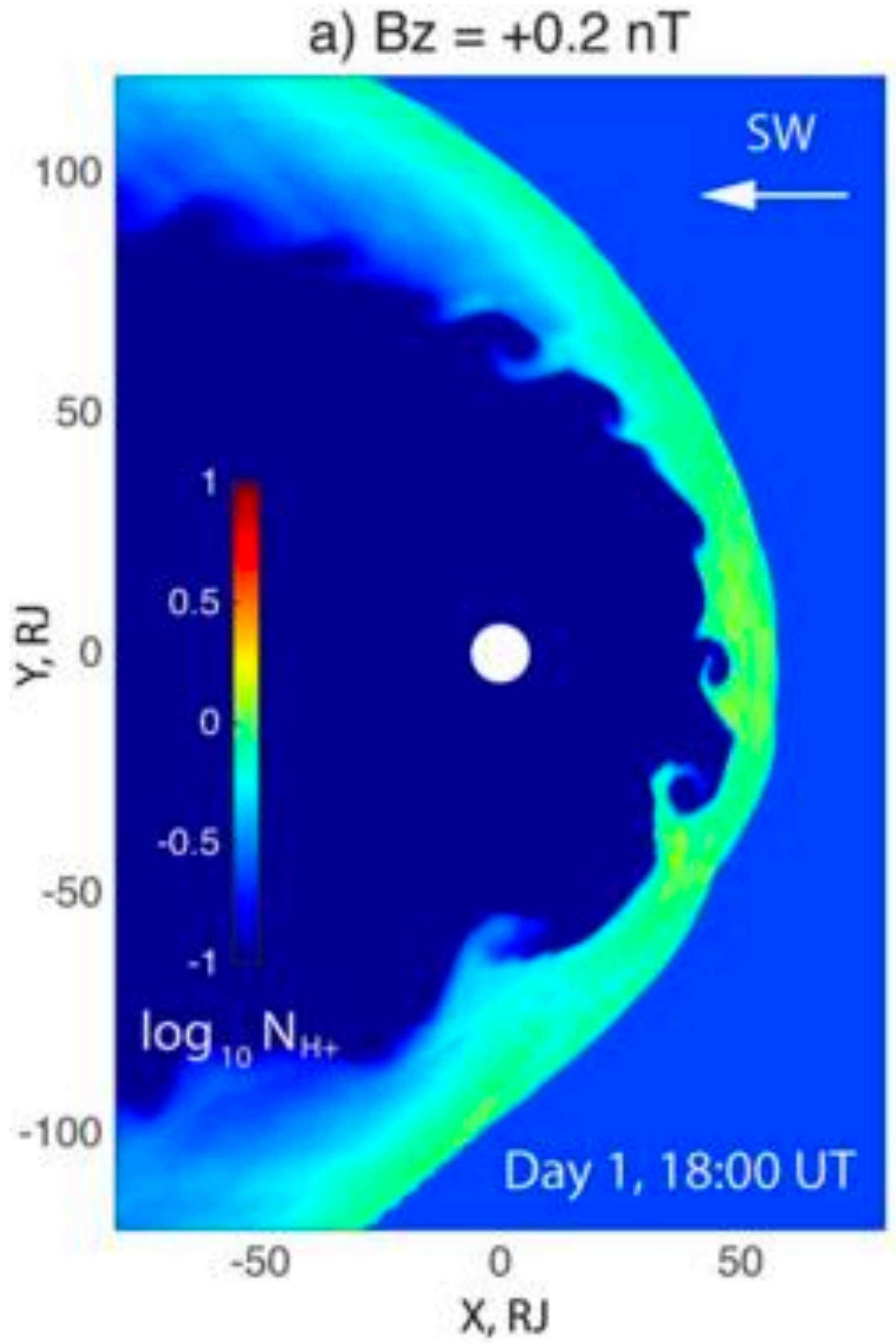
Thermal instabilities for coronal rain



Coronal Mass Ejections can be triggered by instabilities - NASA



The violent release of energy during solar flares can be initiated by instability - NASA



KHI in Jovian magnetosphere Zhang et al. 2017

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Want to find a relationship between the growth rate, ω, and the wave number, **k**, of the perturbation.

$\omega^2 > 0$ **Waves**

Forces oppose any displacement from the equilibrium, creating oscillatory behaviour.

$\omega^2 < 0$ **Instabilities**

Forces enhance any displacement from the equilibrium, creating runaway, exponential growth.

Sausage Instability

Instabilities in a tube

Consider azimuthal magnetic field:

$$\mathbf{B} = B_\phi(R)\hat{\mathbf{e}}_\phi$$

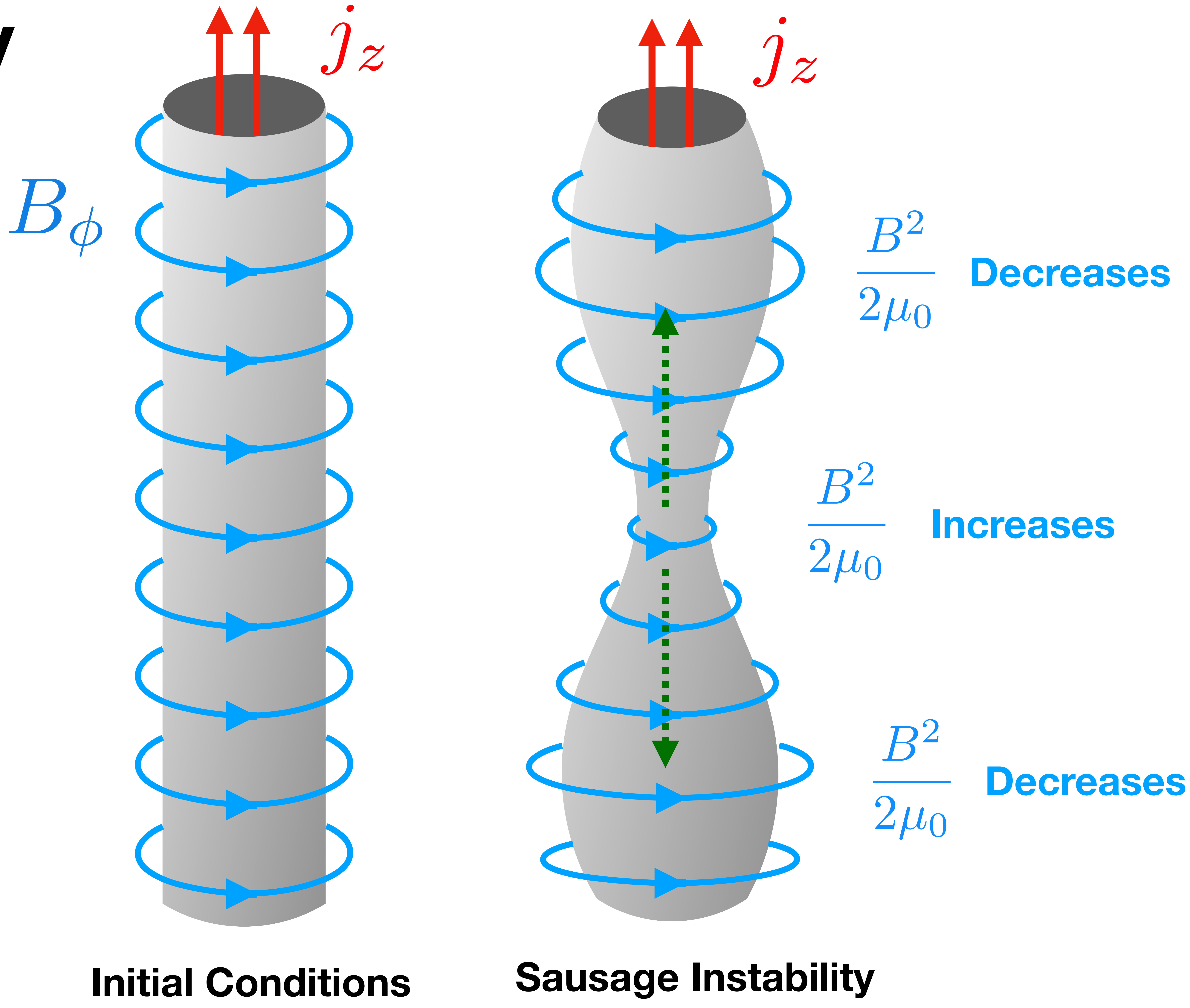
with axial current: $\mathbf{j} = j_z(R)\hat{\mathbf{e}}_z$

Initial equilibrium (no gravity):

$$\mathbf{j} \times \mathbf{B} = \nabla P$$

Magnetic tension balancing gas pressure force.

What happens if the tube is subject to a small pinch?



Kink Instability

Same Initial Conditions:

Consider azimuthal magnetic field:

$$\mathbf{B} = B_\phi(R)\hat{\mathbf{e}}_\phi$$

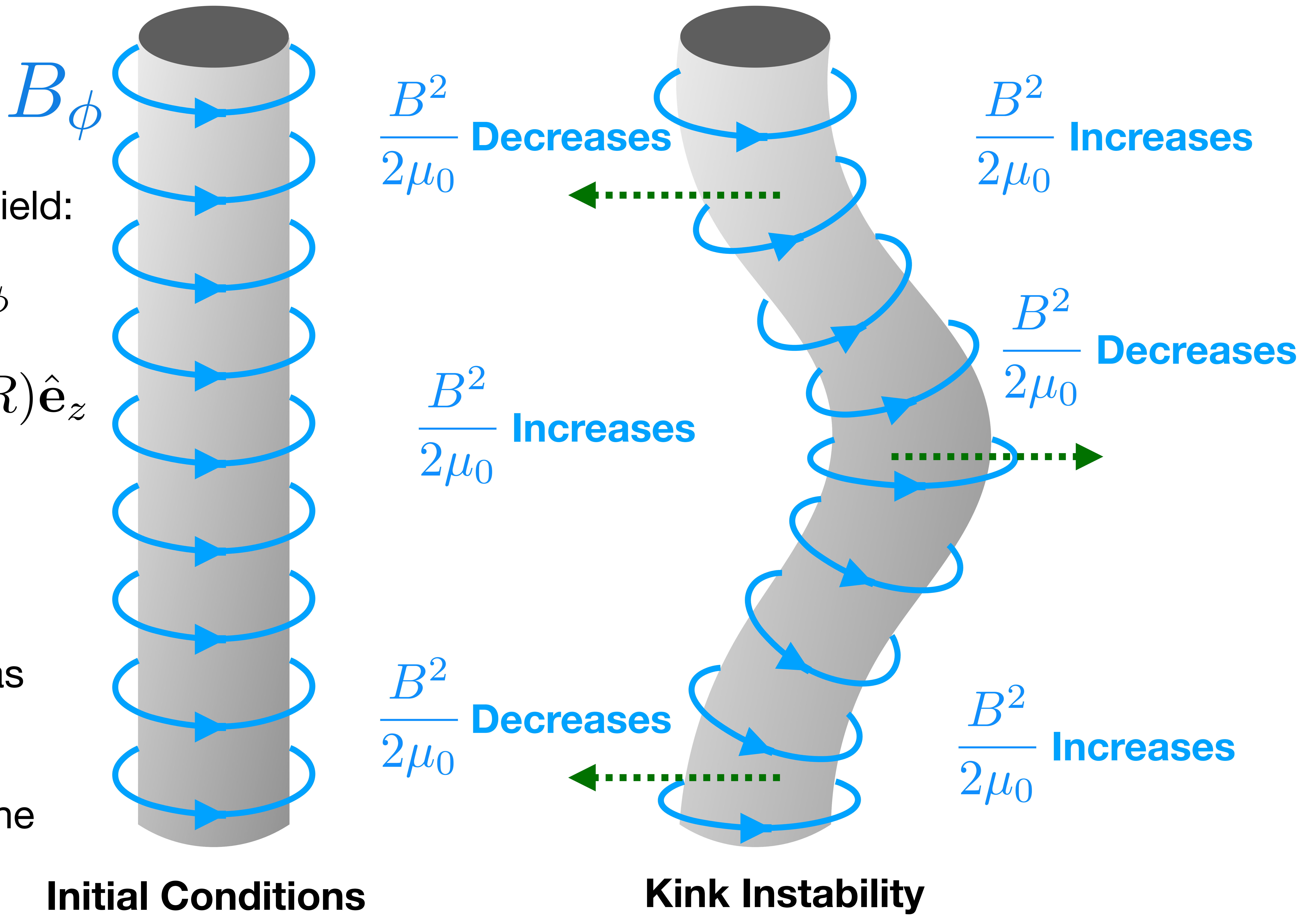
with axial current: $\mathbf{j} = j_z(R)\hat{\mathbf{e}}_z$

Initial equilibrium (no gravity):

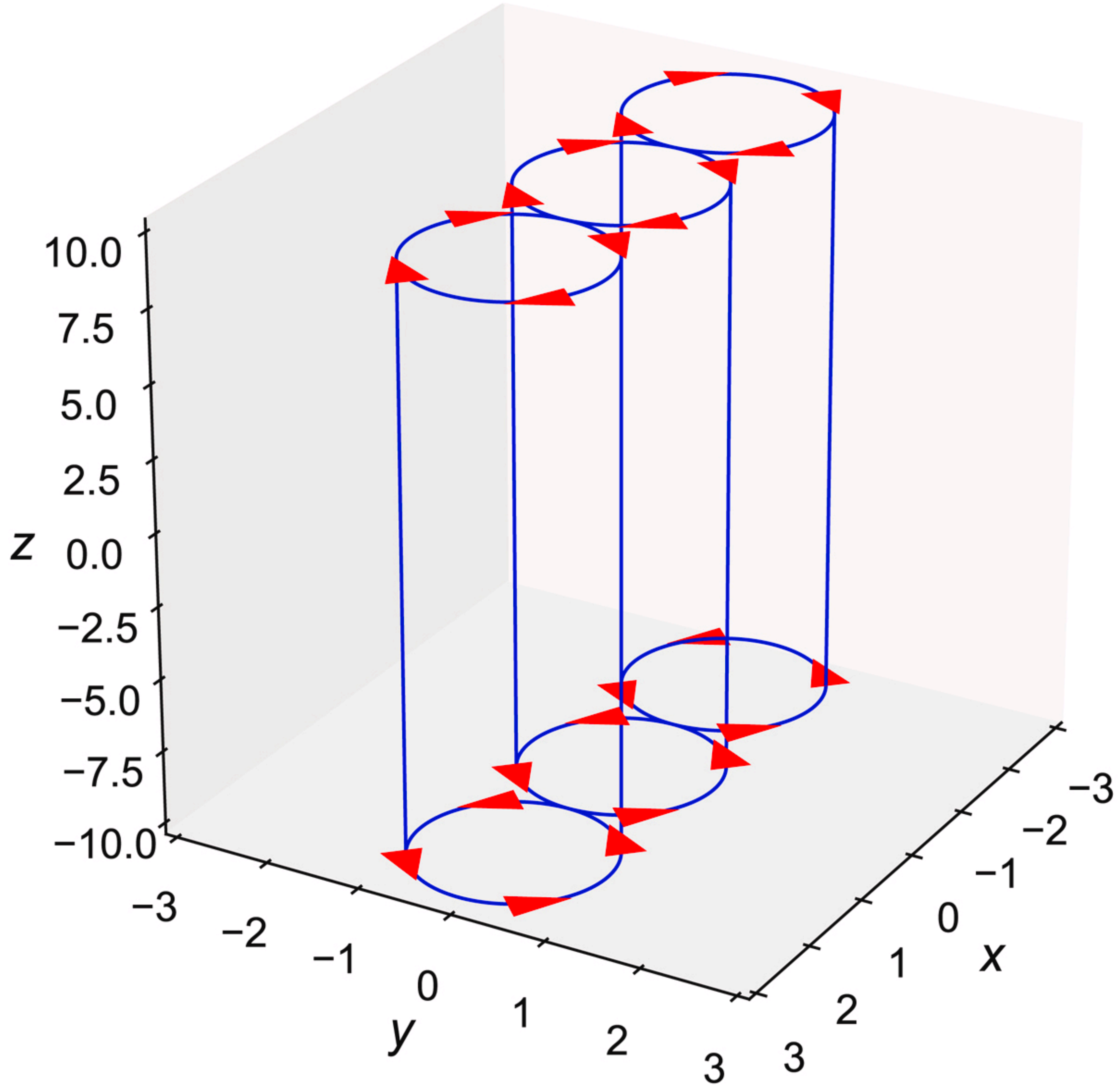
$$\mathbf{j} \times \mathbf{B} = \nabla P$$

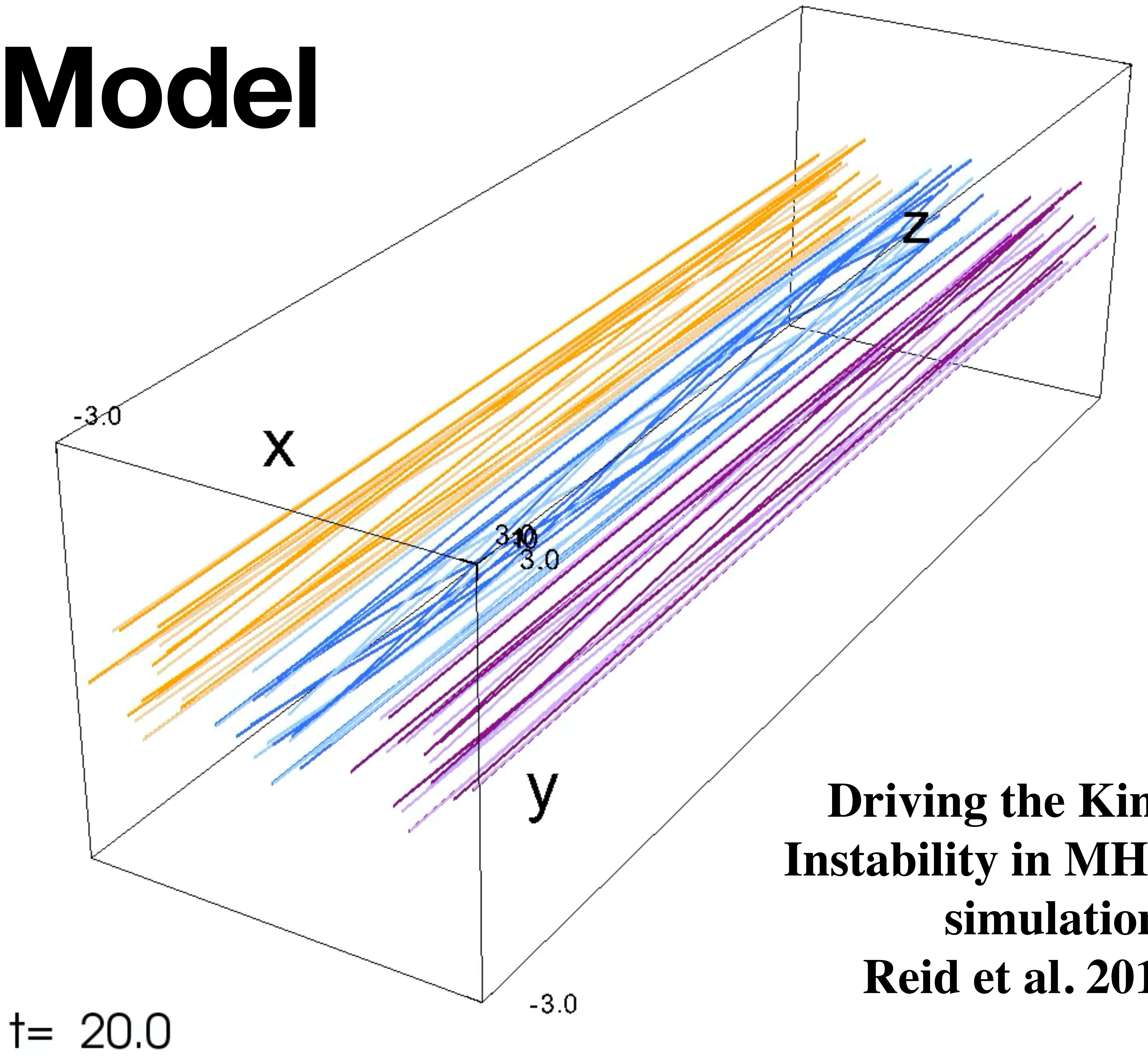
Magnetic tension balancing gas pressure force.

What happens if a section of the tube axis is displaced?



Kink Instability - Numerical Model





Driving the Kink Instability in MHD simulations
Reid et al. 2018

Impose 3 pairs of counter-rotational drivers at the foot points of magnetic field lines.

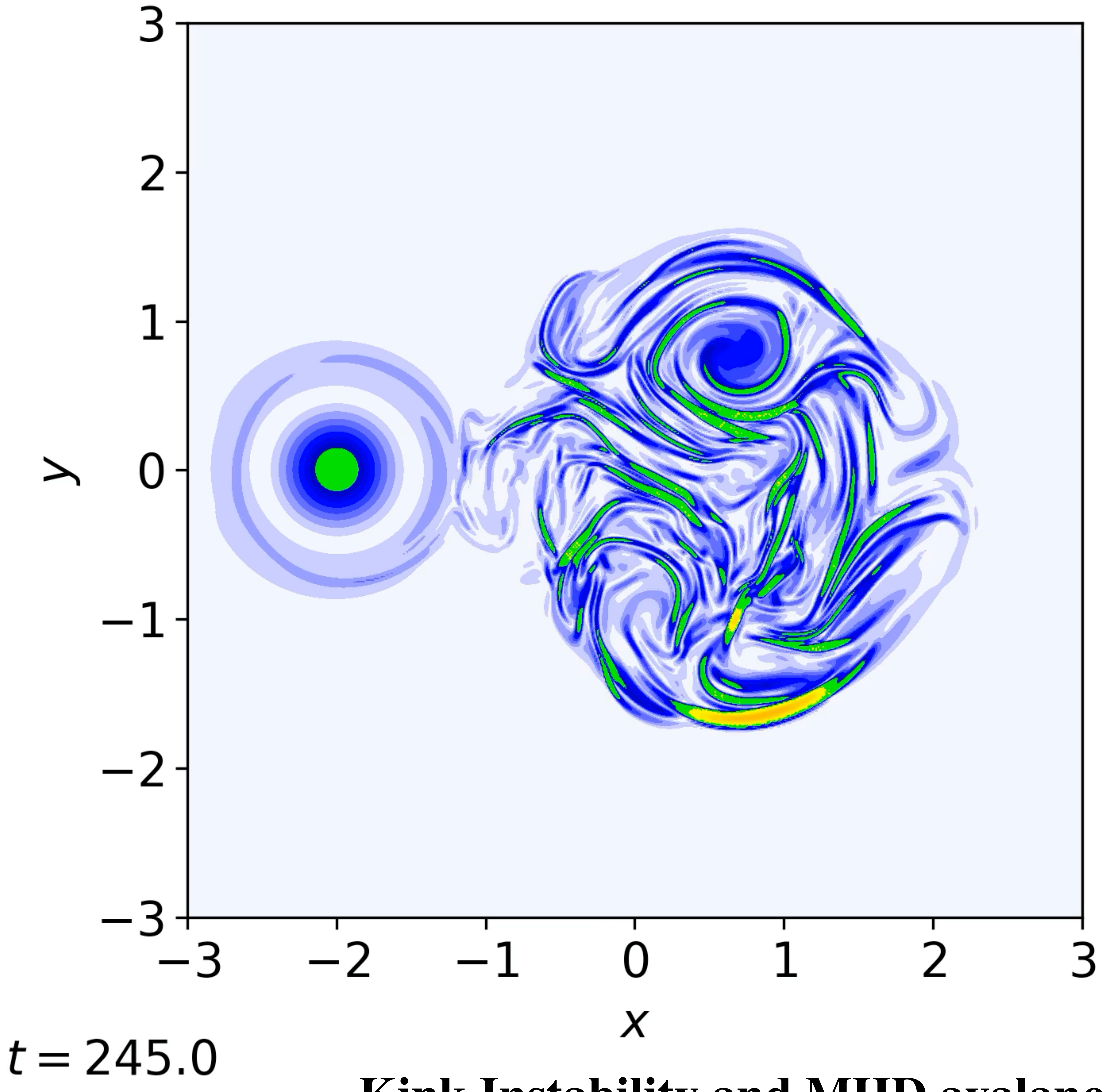
These generate **twisted magnetic flux tubes**. Twist central flux tube at faster rate. Keep rotating and trigger kink instability. What happens next? How can this drive energy release in the corona?

Kink Instability

Kink instability in central flux tube generates small scales throughout flux tube —> **magnetic reconnection, energy release and heating.**

Instability in the central flux tube destabilises neighbouring flux tubes.

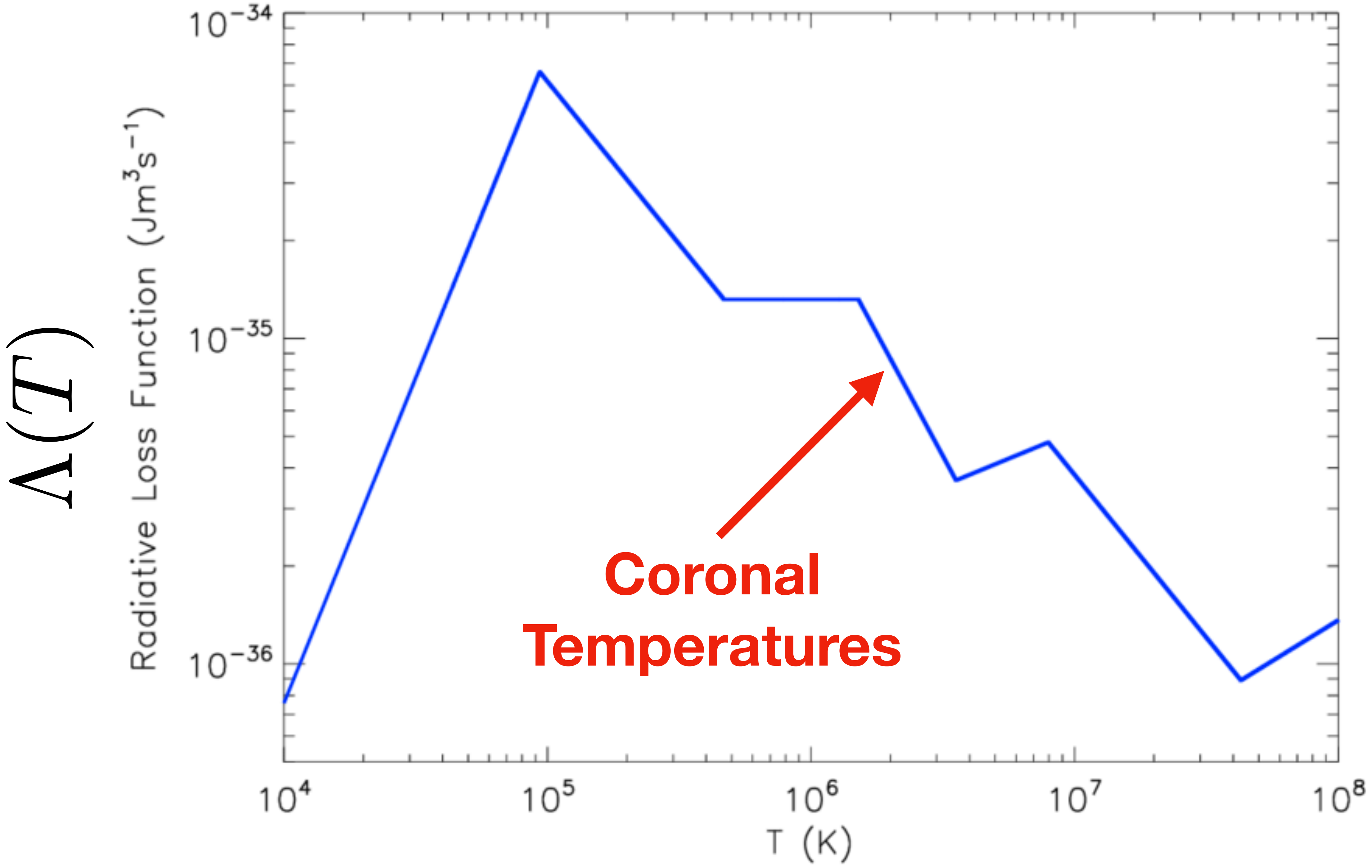
This is an example of an **MHD avalanche** and can explain how one energy release event can trigger many more.



Kink Instability and MHD avalanches
Reid et al. 2018

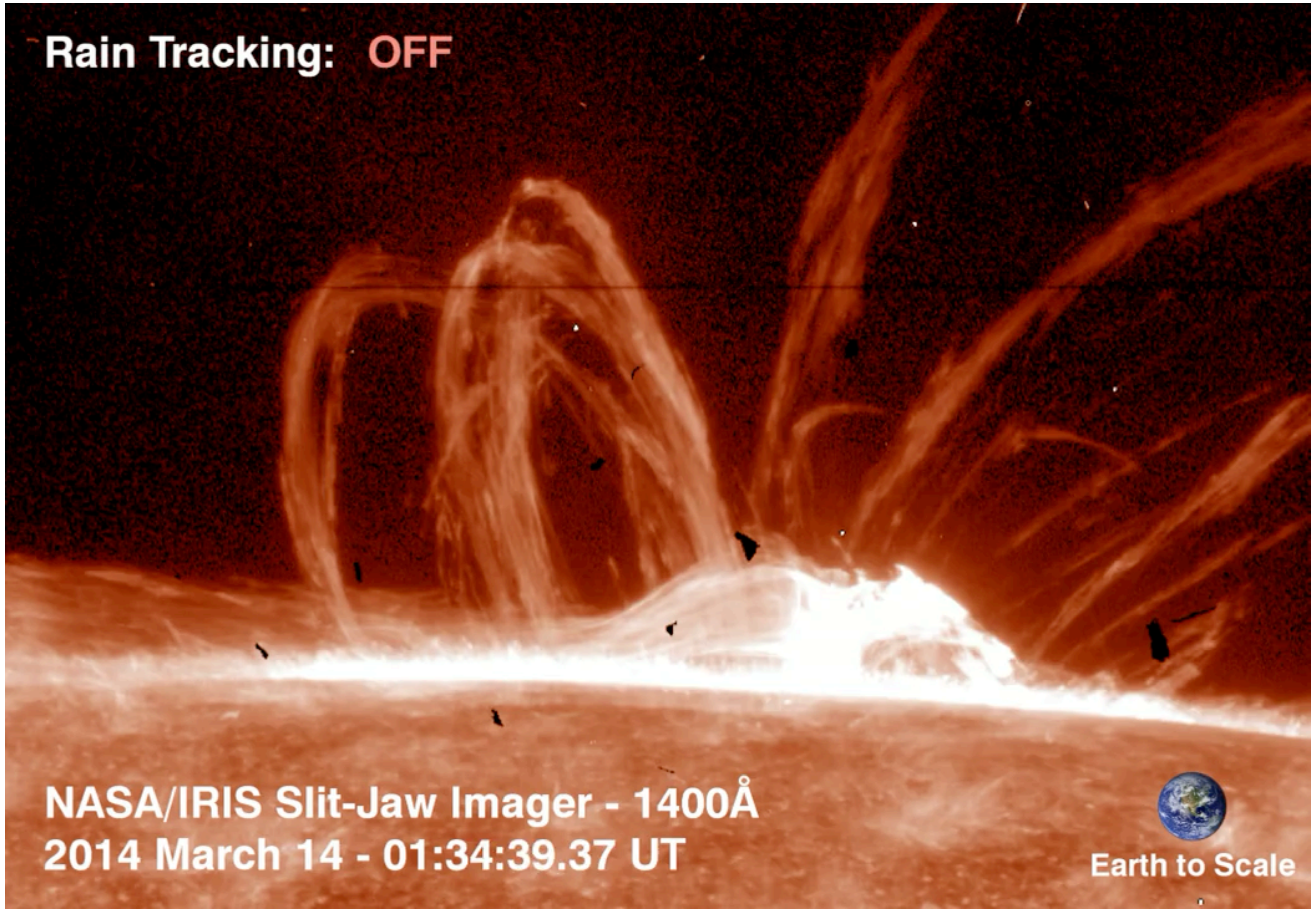
Thermal Instability

Optically Thin Radiative Losses
(Klimchuk et al. 2008)



$$\frac{\mathrm{d}\Lambda}{\mathrm{d}T} > 0$$ **Radiatively Stable**

$$\frac{\mathrm{d}\Lambda}{\mathrm{d}T} < 0$$ **Radiatively Unstable**



1. Thermal conduction typically dominant stabilising term in the corona.
2. If not, then thermal instability leads to cooling
3. Pressure decreases, so attracts more plasma due to pressure forces.
4. This leads to more cooling and process continues until gravity causes condensations to fall as rain.

Rayleigh-Taylor and Kelvin-Helmholtz instabilities

MHD supports standard HD instabilities although the magnetic field can have a stabilising effect due to **magnetic tension.**

Initial Equilibrium

$$\rho_0(z) = \begin{cases} \rho_- & \text{if } z < 0, \\ \rho_+ & \text{if } z \geq 0. \end{cases}$$

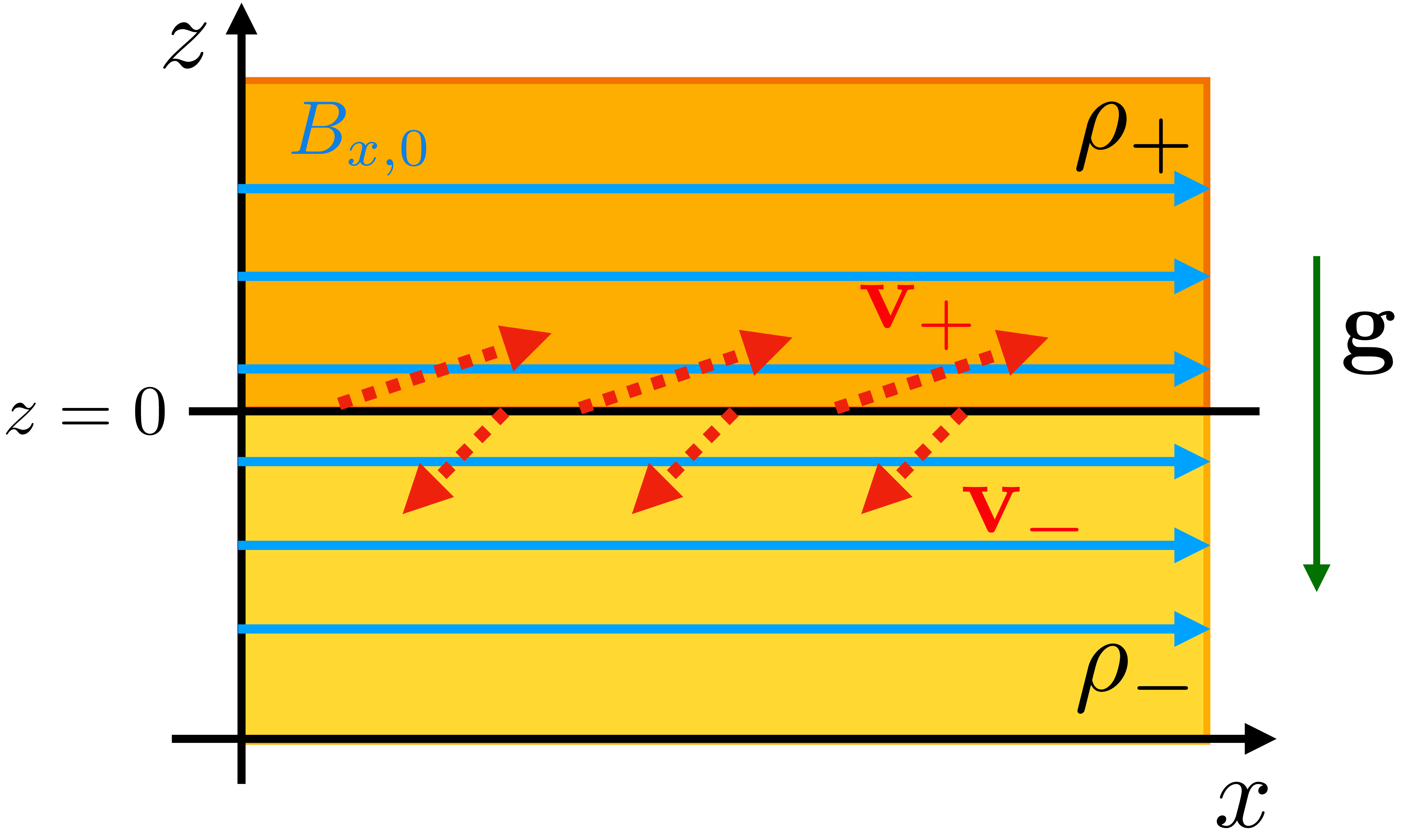
$$P_0(z) = A_0 - \rho_0 g z$$

$$\mathbf{B}_0 = (B_{x,0}, 0, 0)$$

$$v_{x,0}(z) = \begin{cases} v_{x,-} & \text{if } z < 0, \\ v_{x,+} & \text{if } z \geq 0. \end{cases}$$

$$v_{y,0}(z) = \begin{cases} v_{y,-} & \text{if } z < 0, \\ v_{y,+} & \text{if } z \geq 0. \end{cases}$$

$$v_z = 0$$



Rayleigh-Taylor and Kelvin-Helmholtz instabilities

MHD supports standard HD instabilities although the magnetic field ... **nsion.**

Initia...

$\rho_0(z) = \begin{cases} \ldots \end{cases}$

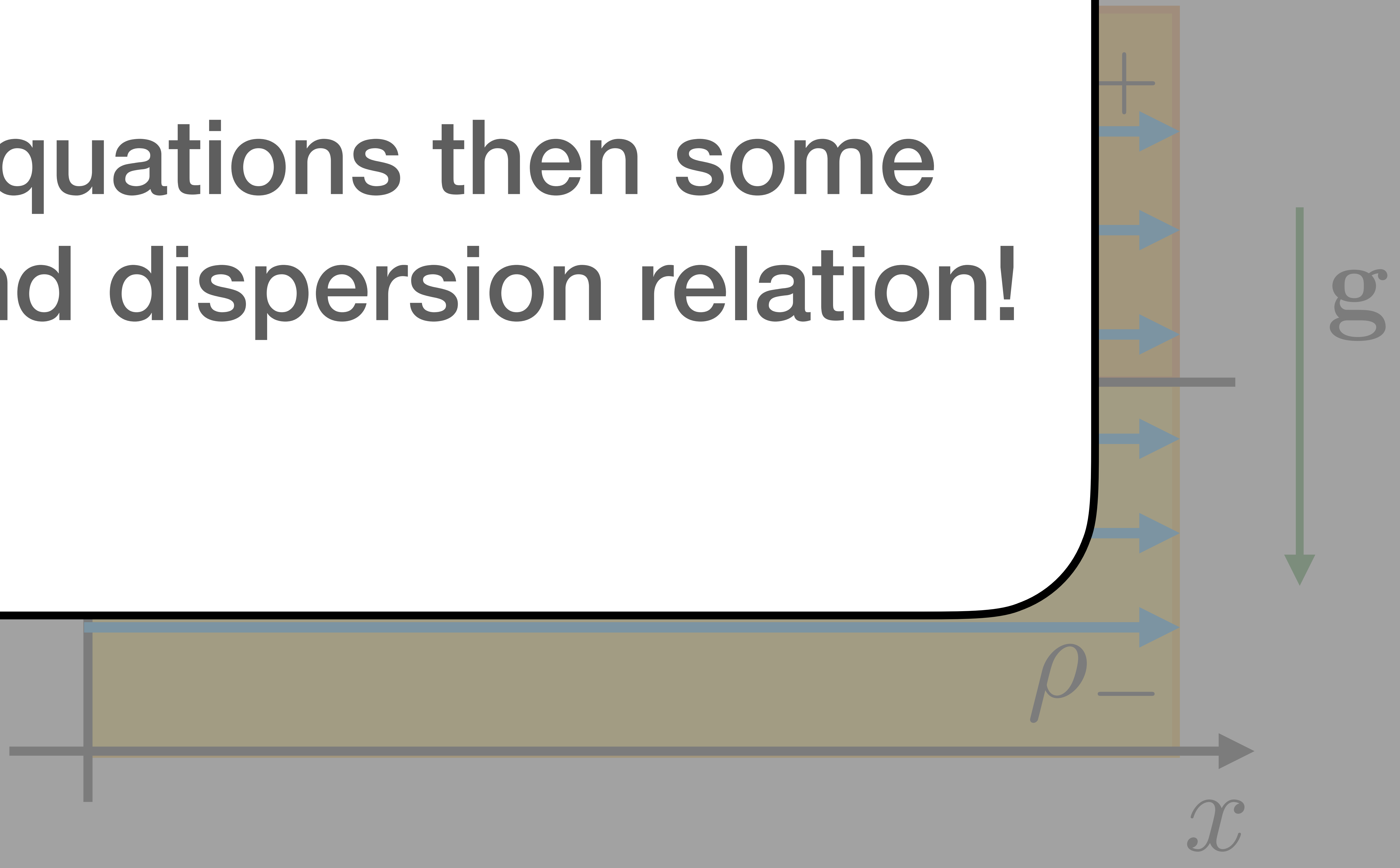
$P_0(z) = A_0 \ldots$

$\mathbf{B}_0 = (B_{x,0} \ldots$

$v_{x,0}(z) = \begin{cases} \ldots \end{cases}$

$$v_{y,0}(z) = \begin{cases} v_{y,-} & \text{if } z < 0, \\ v_{y,+} & \text{if } z \geq 0. \end{cases}$$

$v_z = 0$



Linearise equations then some algebra to find dispersion relation!

Rayleigh-Taylor and Kelvin-Helmholtz instabilities

$$\alpha_+ = \frac{\rho_+}{\rho_+ + \rho_-} \qquad \alpha_- = \frac{\rho_-}{\rho_+ + \rho_-} \qquad \mathbf{\Delta v} = \mathbf{v}_+ - \mathbf{v}_-$$

$$\omega^* = \boxed{-\left(\alpha_+ \left(\mathbf{k} \cdot \mathbf{v}_+\right) + \alpha_- \left(\mathbf{k} \cdot \mathbf{v}_-\right)\right)} \pm \sqrt{\boxed{2\alpha_- \left(\mathbf{k} \cdot \mathbf{B}_0\right)^2} \boxed{- gk \left(\alpha_+ - \alpha_-\right)} \boxed{- \alpha_+ \alpha_- \left(\mathbf{k} \cdot \mathbf{\Delta v}\right)^2}}$$

1. Advection of oscillatory solution through observer's rest frame.

Term under the square root determines stability. **Stable solutions if > 0, unstable solutions if < 0.**

2. Positive term suppresses instability due to **magnetic tension.**

3. Negative term drives instability **(Rayleigh-Taylor instability).**

4. Negative term drives instability **(Kelvin-Helmholtz instability).**

*Expression has been simplified by normalising density and Alfvén speed of lower region.

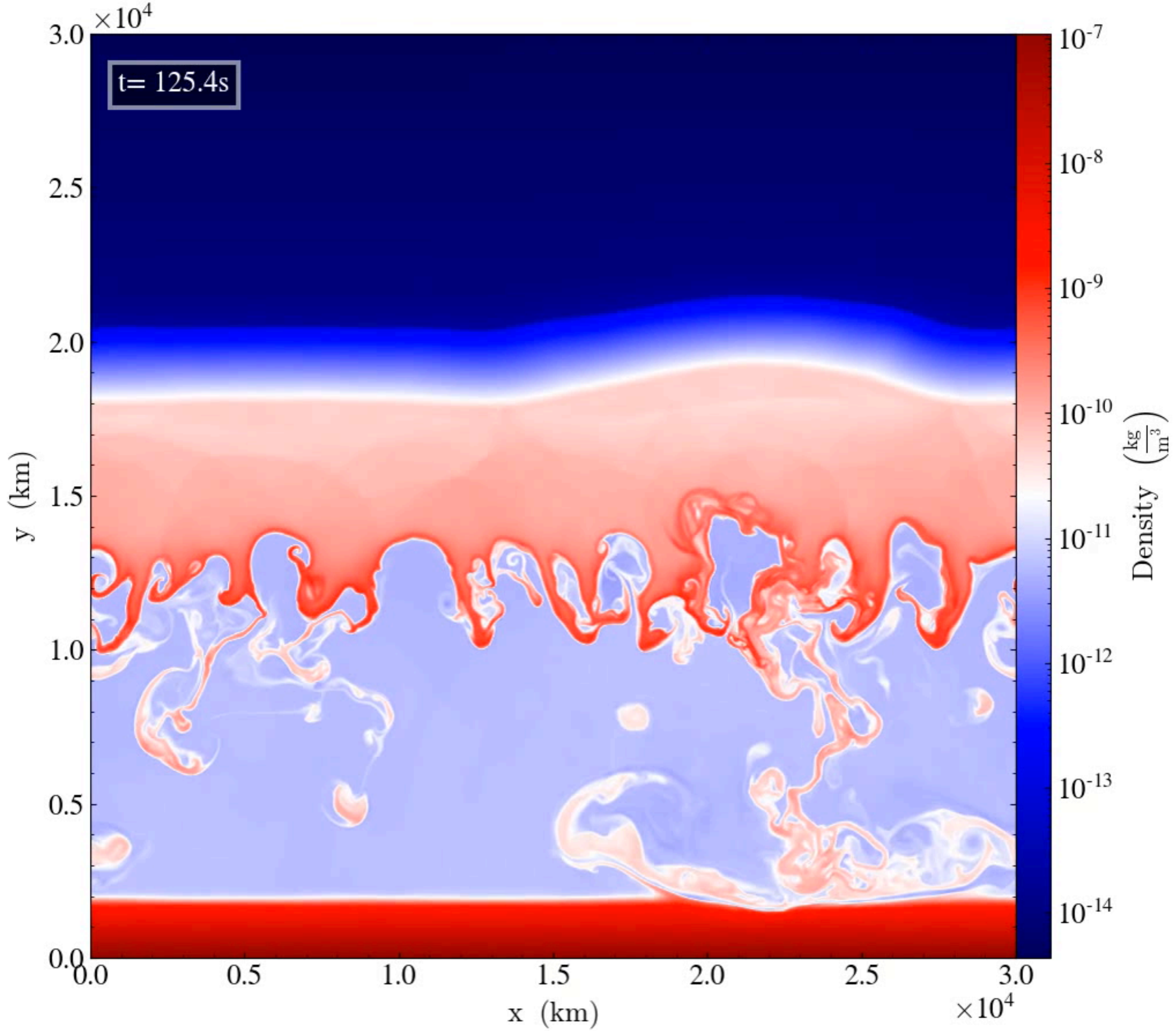
Rayleigh-Taylor

Hydrodynamic instability which forms when a high density fluid lies above a lower density fluid

Modified in MHD, in particular by magnetic tension force

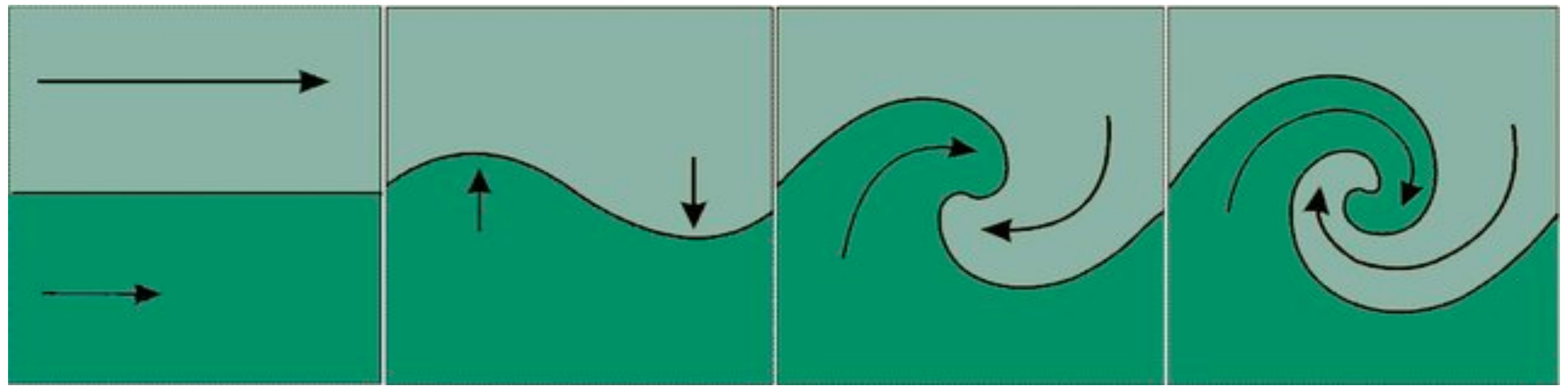
Well-studied in solar prominences.

As with many instabilities, growth rate reduced by non-ideal effects such as viscosity.



MHD Simulation of Rayleigh-Taylor Instability in a Prominence
Changmai et al. 2023

Kelvin-Helmholtz instability



Kelvin-Helmholtz Instability
Philippi et al. 2015

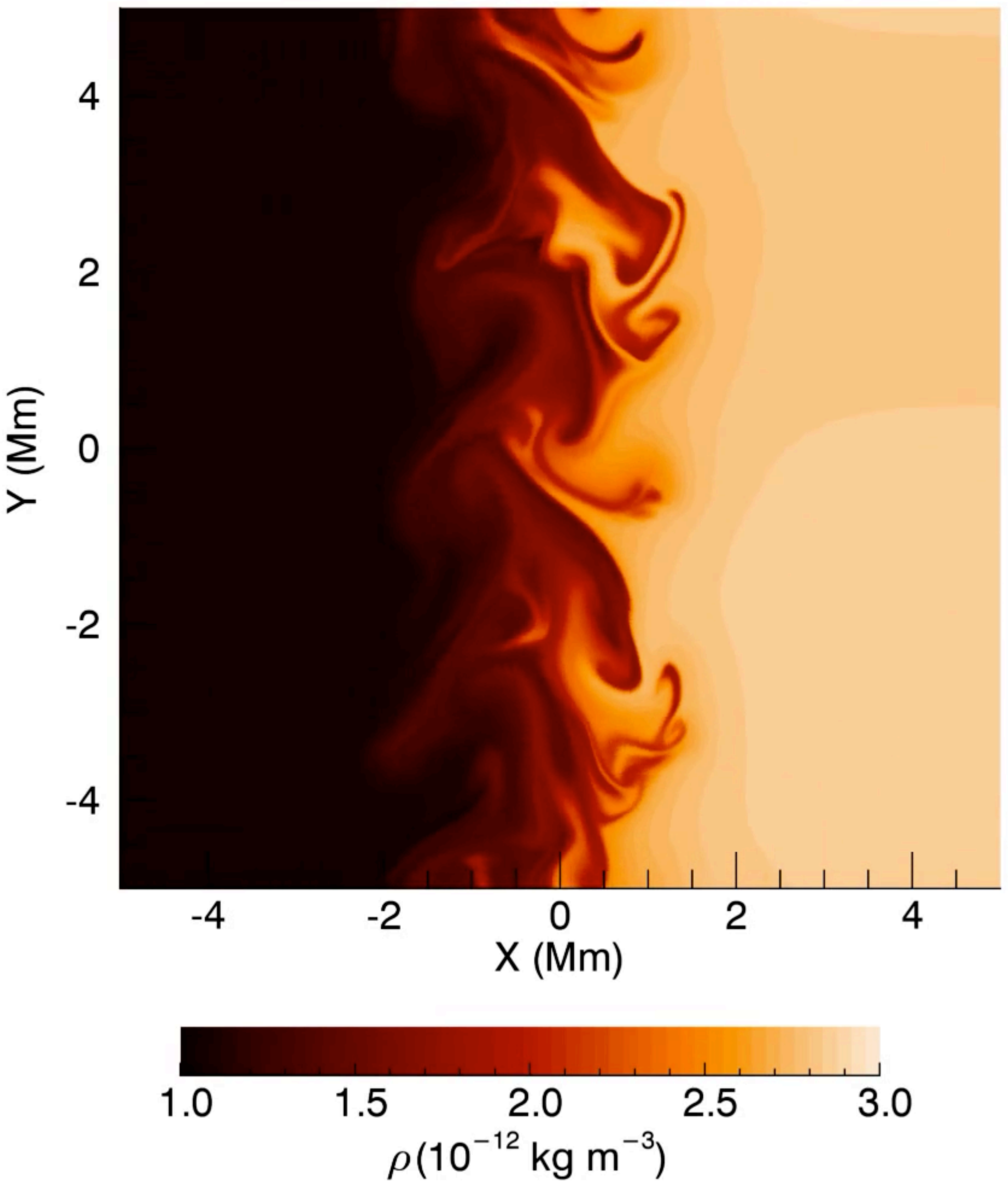
Hydrodynamic instability driven by a velocity shear across an interface.

Again, modified in MHD, in particular by magnetic tension force

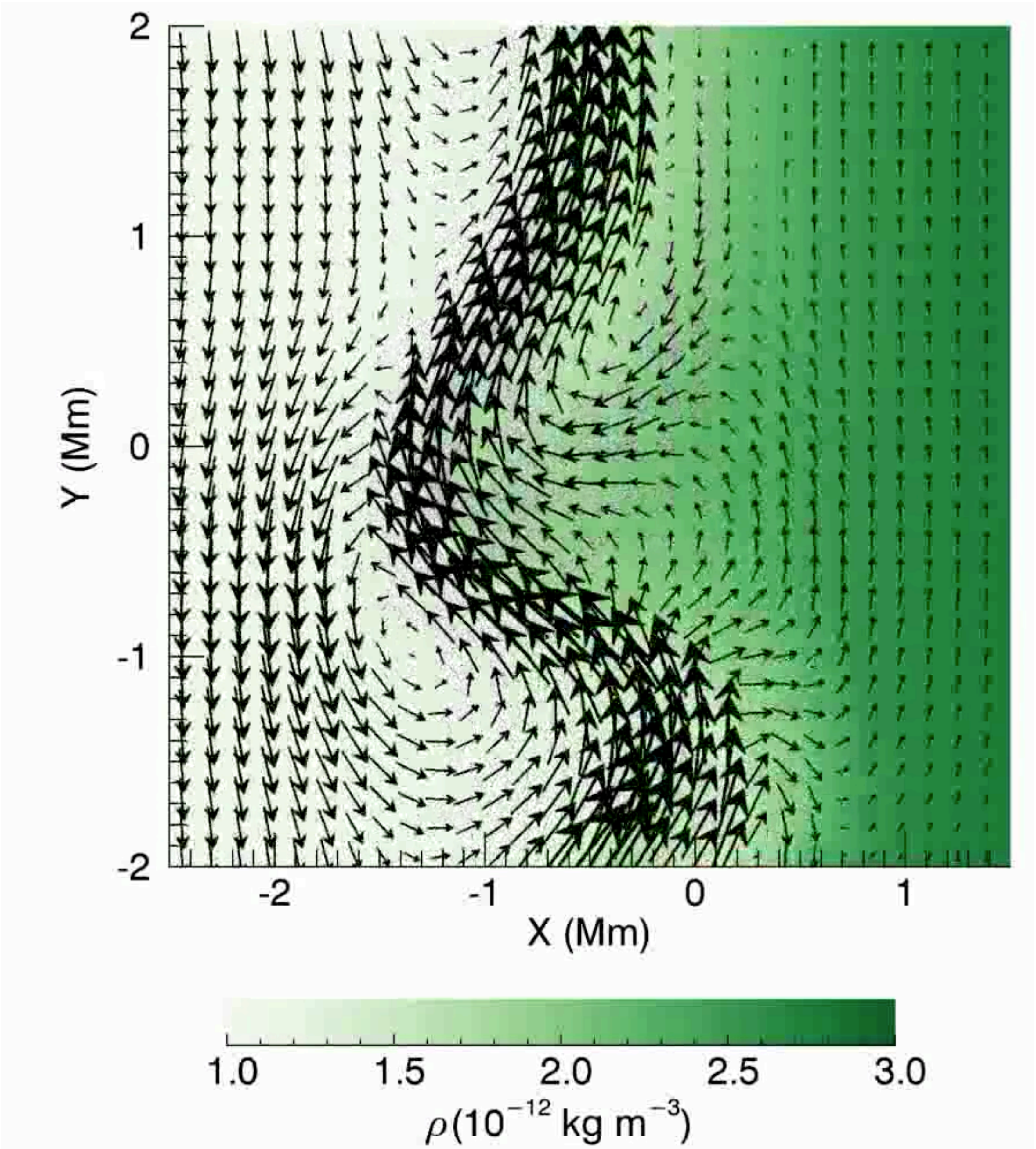
Can develop with an oscillating velocity shear, e.g. due to out-of-phase Alfvén waves.

Growth rate reduced or suppressed by sheared magnetic field across interface.

Density

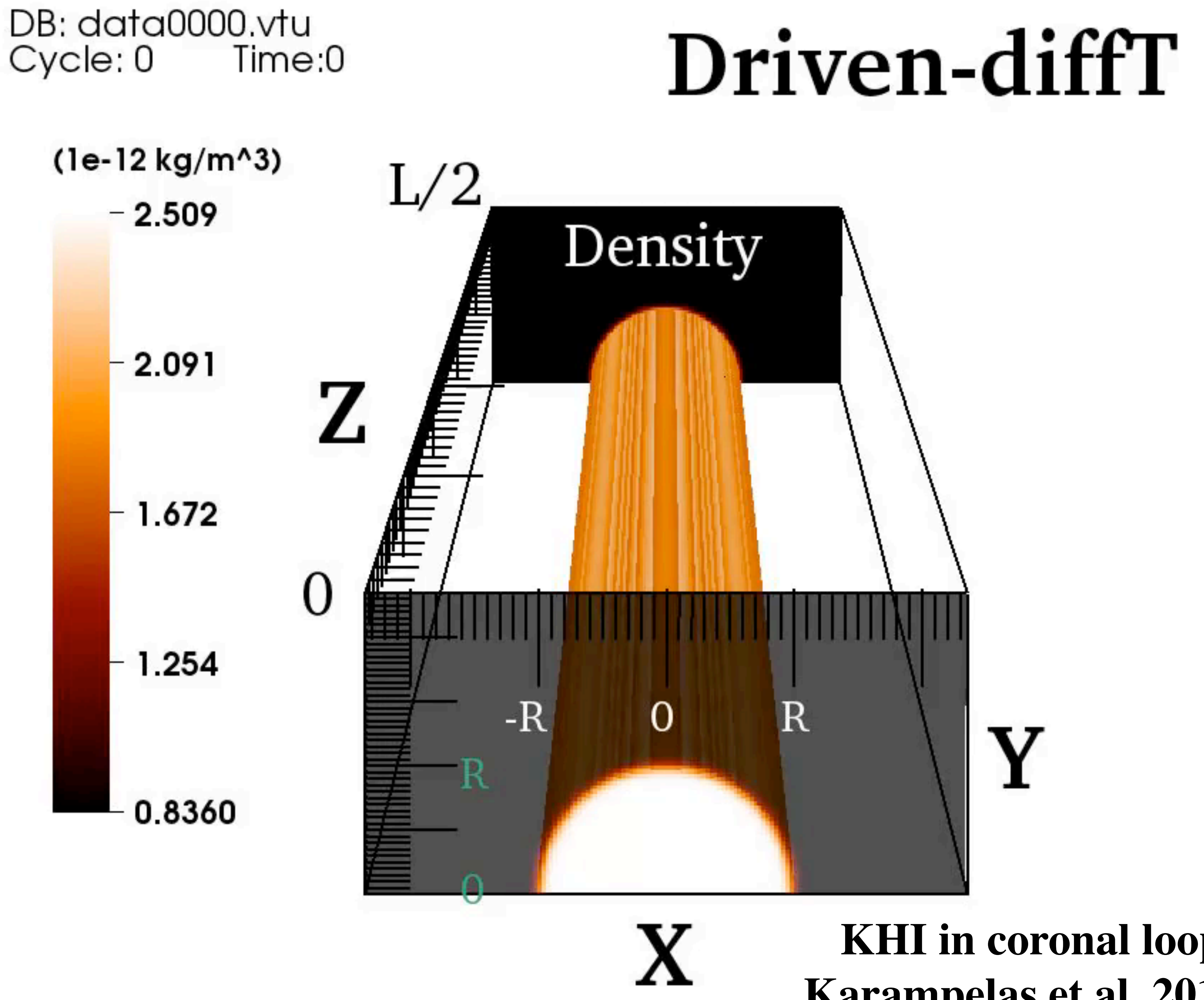


Density and Velocity

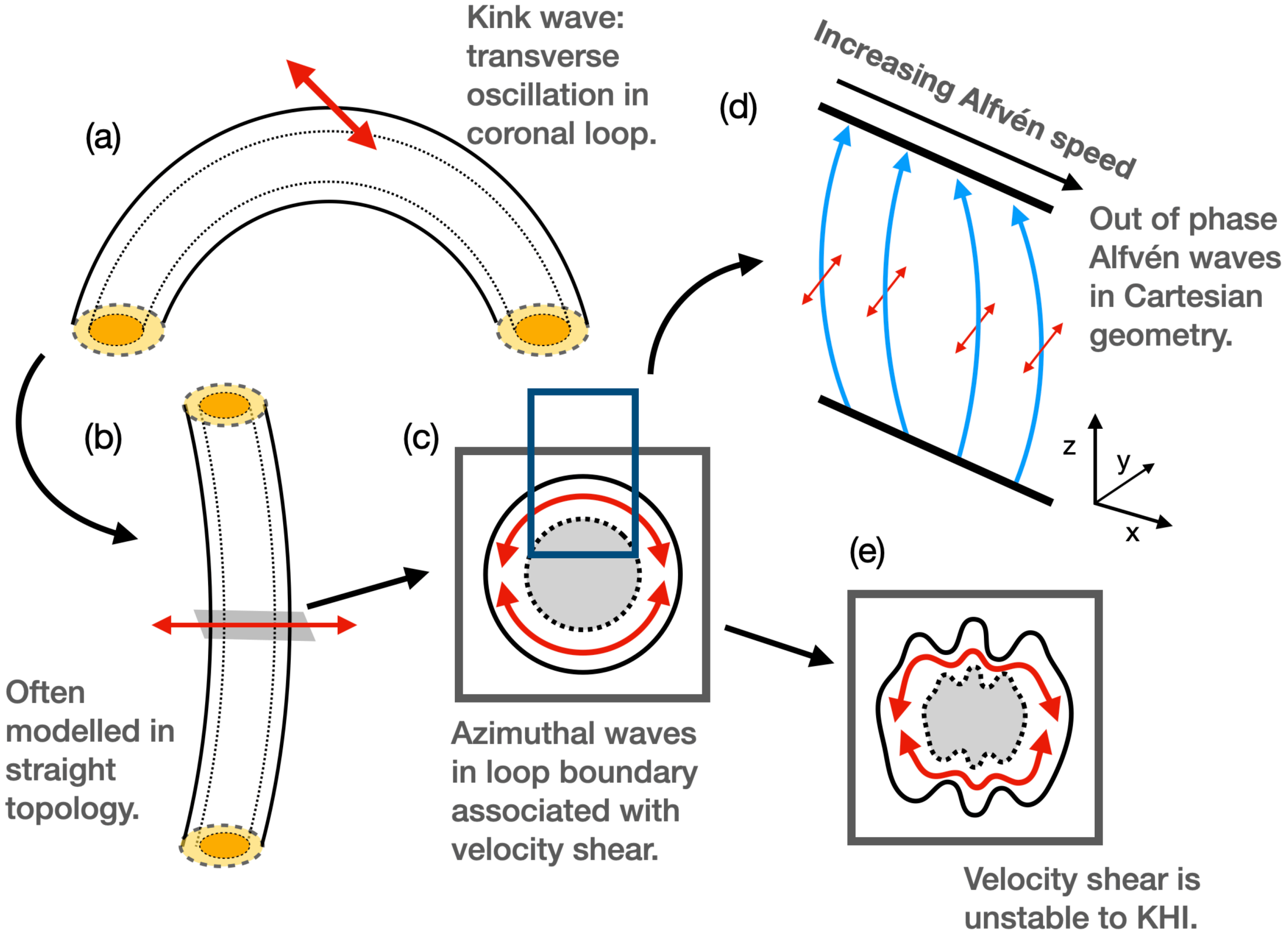


KHI in Coronal Loops

1. Foot point driving excites kink mode.
2. Correct frequency -> fundamental standing wave.
3. Energy transferred to azimuthal Alfvén mode.
4. Radial gradients are unstable to KHI.



KHI in coronal loops
Karampelas et al. 2017



Instability growth rates are reduced in **short flux tubes** (due to magnetic tension), **dissipative regimes** (removes energy from the shear flow) and for **twisted magnetic fields**.

Concluding Remarks

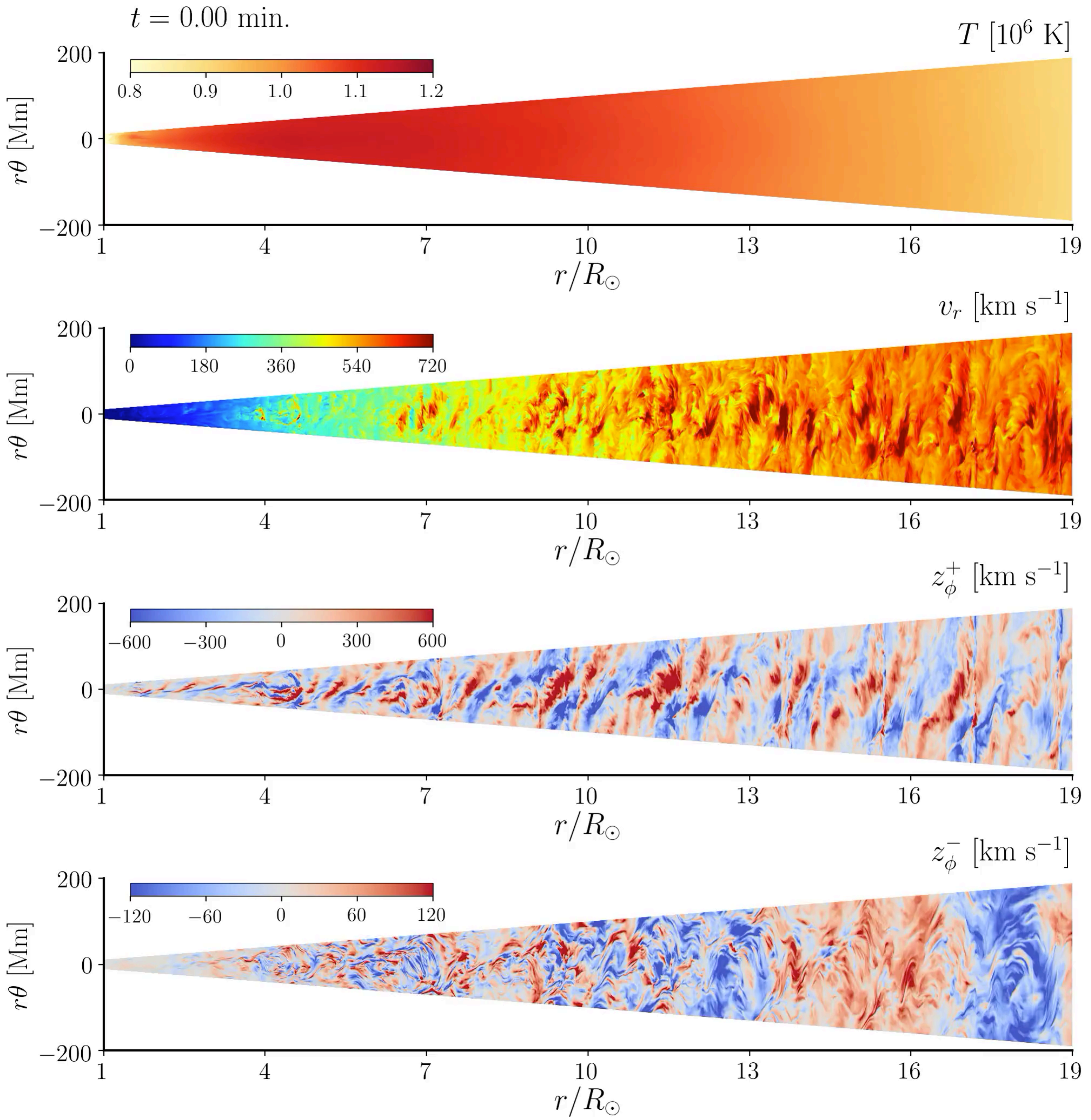
The inclusion of a magnetic field in MHD creates a wide range of new dynamics in comparison to HD systems.

This includes the magnetic field introducing **anisotropies**.

Both the **magnetic tension force** and the **magnetic pressure force** are important for understanding waves and instabilities in magnetised plasmas.

Linearisation is a powerful tool for understanding the stability of physical systems and describing the response to small perturbations.

However the Universe is **non-linear** and non-linear evolution can be important (e.g. instabilities can grow very quickly!). Can still make progress though with more sophisticated analytical techniques or numerical models!



Modelling oscillations in the solar wind
Shoda et al. 2019