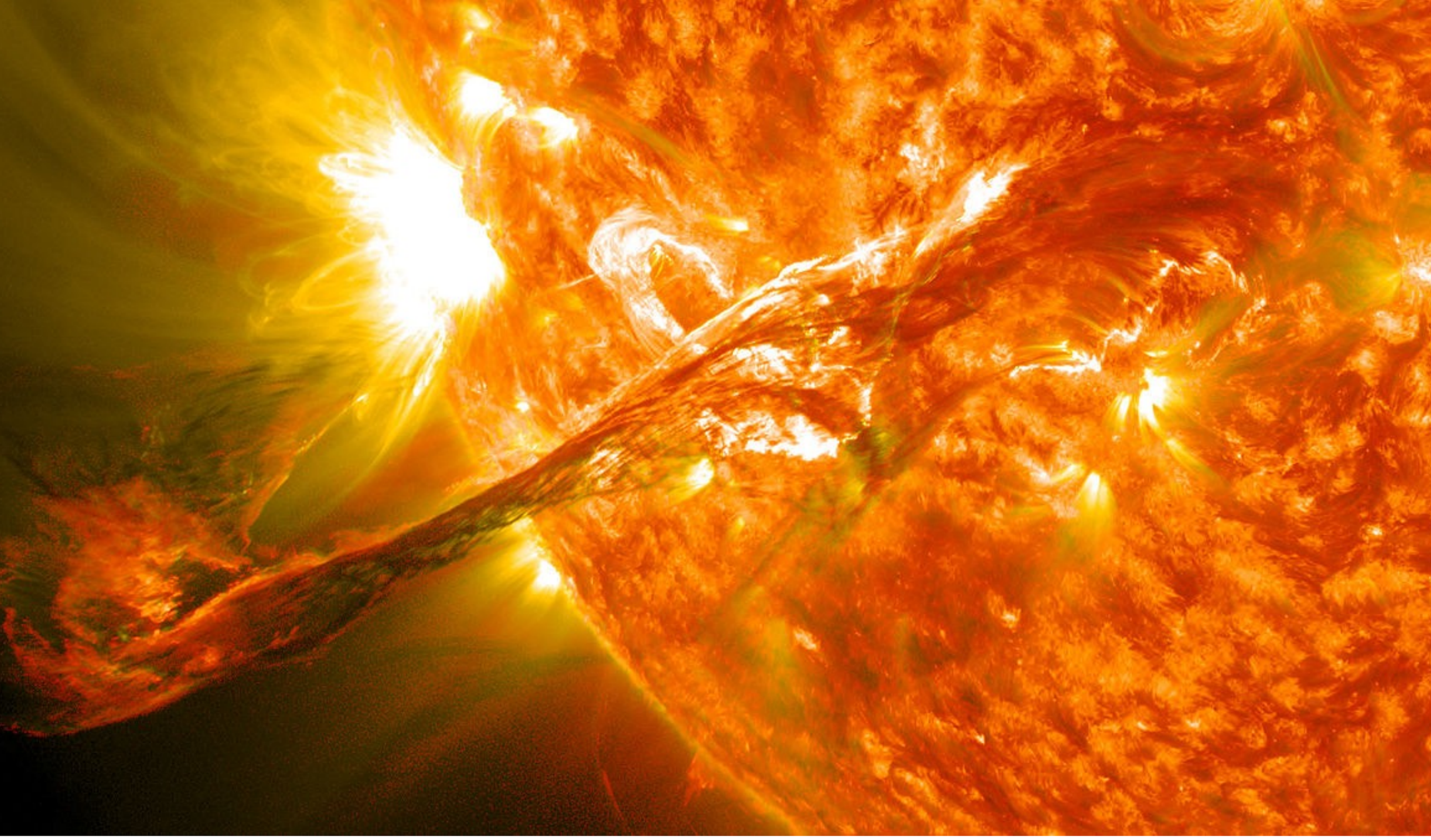
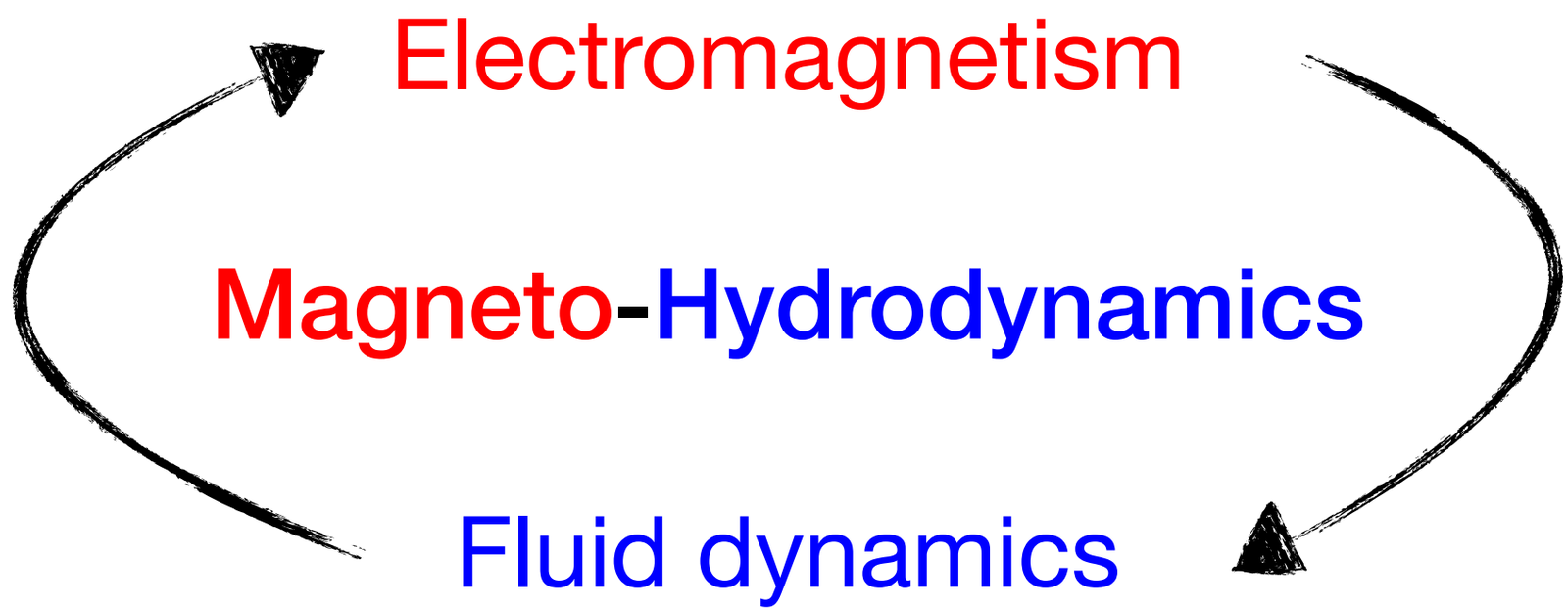


Introduction to Magnetohydrodynamics

Dr Alexander Russell
University of St Andrews


Electromagnetism
Magneto-Hydrodynamics
Fluid dynamics

Birth of MHD

Prof. Hannes Alfvén
30 May 1908 - 2 April 1995
Education: Uppsala 1927-1934
Advisor: M. Siegbahn

Existence of Electromagnetic-Hydrodynamic Waves

If a conducting liquid is placed in a constant magnetic field, every motion of the liquid gives rise to an E.M.F. which produces electric currents. Owing to the magnetic field, these currents give mechanical forces which change the state of motion of the liquid.

Thus a kind of combined electromagnetic-hydrodynamic wave is produced which, so far as I know, has as yet attracted no attention.

The phenomenon may be described by the electrodynamic equations

$$\text{rot } H = \frac{4\pi}{c} i$$

$$\text{rot } E = -\frac{1}{c}\frac{dB}{dt}$$

$$B = \mu H$$

$$i = \sigma(E + \frac{v}{c} \times B);$$

together with the hydrodynamic equation

$$\partial \frac{dv}{dt} = \frac{1}{c}(i \times B) - \text{grad } p,$$

where σ is the electric conductivity, μ the permeability, ∂ the mass density of the liquid, i the electric current, v the velocity of the liquid, and p the pressure.

Consider the simple case when $\sigma = \infty$, $\mu = 1$ and the imposed constant magnetic field H_0 is homogeneous and parallel to the z-axis. In order to study a plane wave we assume that all variables depend upon the time t and z only. If the velocity v is parallel to the x-axis, the current i is parallel to the y-axis and produces a variable magnetic field H' in the x-direction. By elementary calculation we obtain

$$\frac{d^2H'}{dz^2} = \frac{4\pi\partial}{H_0^2}\frac{d^2H'}{dt^2},$$

which means a wave in the direction of the z-axis with the velocity

$$V = \frac{H_0}{\sqrt{4\pi\partial}}.$$

Waves of this sort may be of importance in solar physics. As the sun has a general magnetic field, and as solar matter is a good conductor, the conditions for the existence of electromagnetic-hydrodynamic waves are satisfied. If in a region of the sun we have $H_0 = 15$ gauss and $\partial = 0{\cdot}005$ gm. cm.$^{-3}$, the velocity of the waves amounts to

$$V \sim 60 \text{ cm. sec.}^{-1}.$$

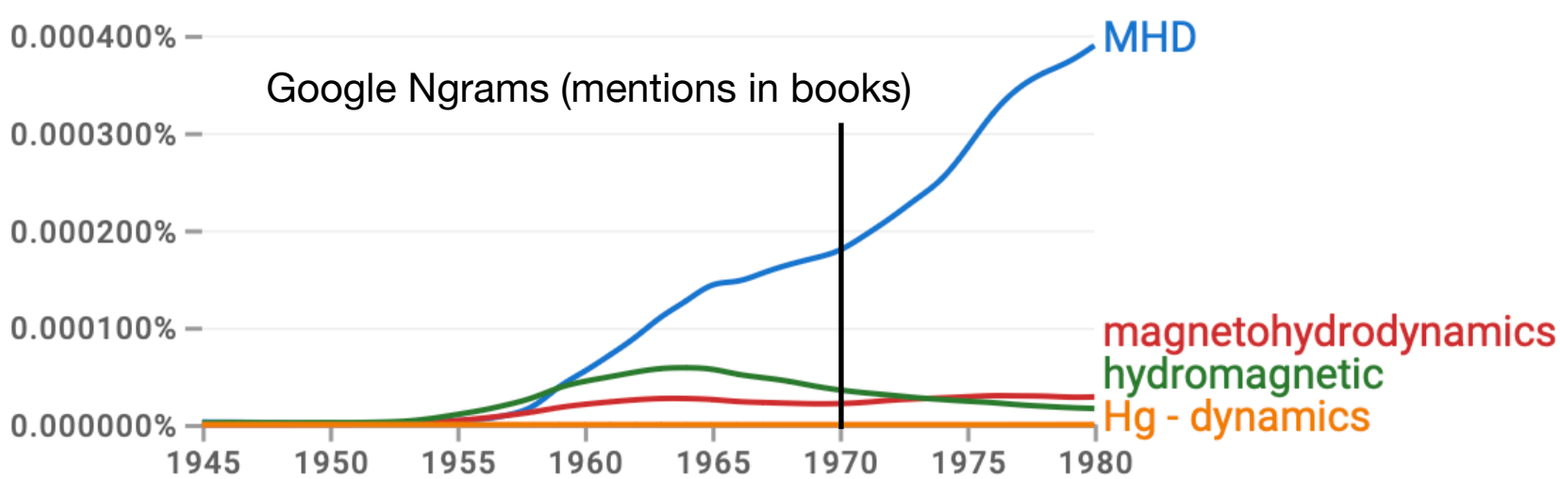
This is about the velocity with which the sunspot zone moves towards the equator during the sunspot cycle. The above values of H_0 and ∂ refer to a distance of about 10^{10} cm. below the solar surface where the original cause of the sunspots may be found. Thus it is possible that the sunspots are associated with a magnetic and mechanical disturbance proceeding as an electromagnetic-hydrodynamic wave.

The matter is further discussed in a paper which will appear in *Arkiv för matematik, astronomi och fysik*.

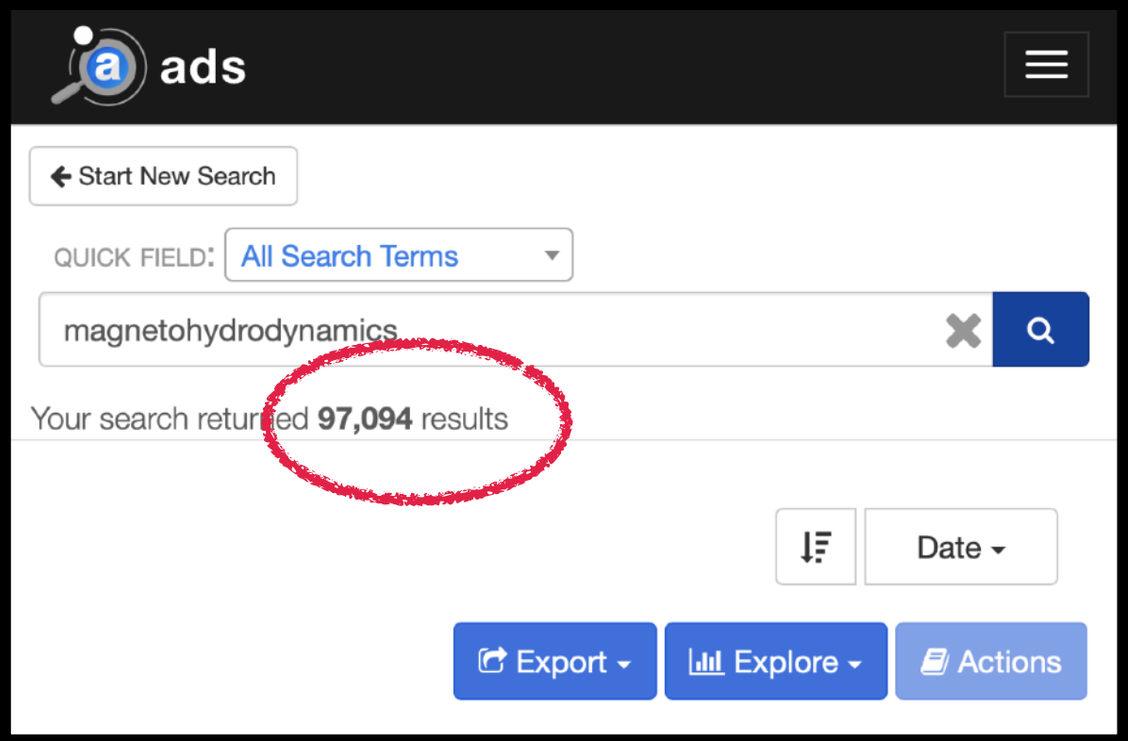
H. Alfvén.

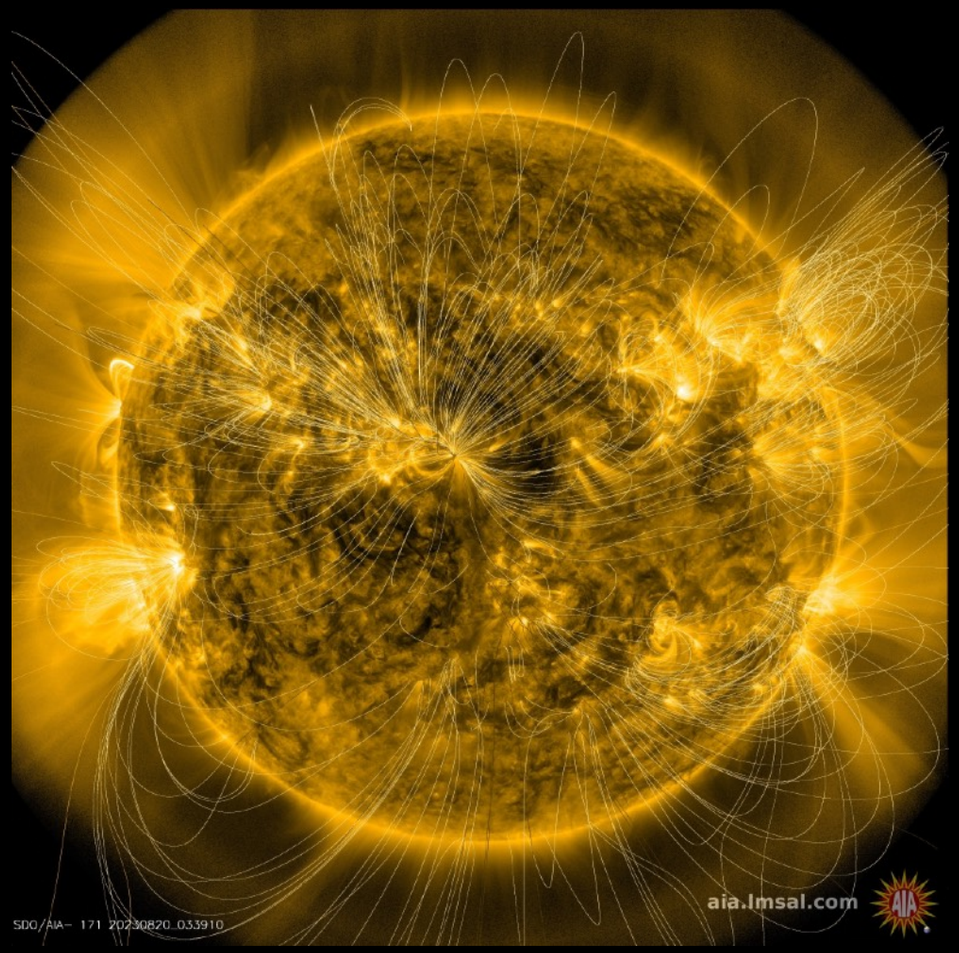
Kgl. Tekniska Högskolan,
Stockholm.
Aug. 24.

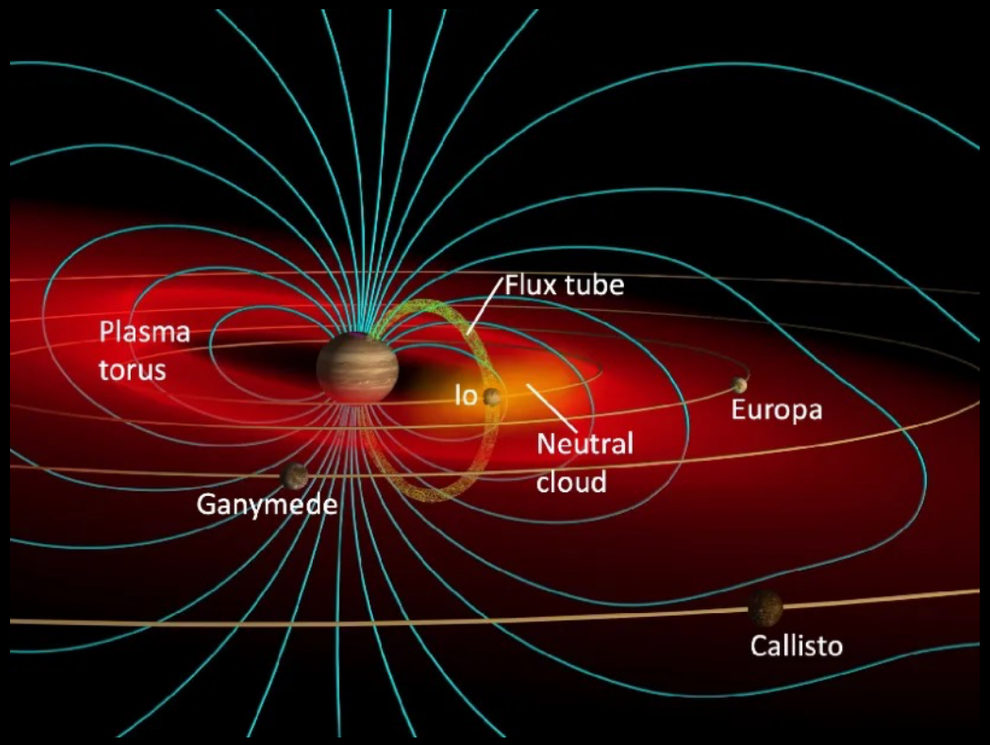
Alfvén (1942), Nature **150**: 405.



Pay Attention Because…







MHD gives you the big picture.

Embodies key physical principles.

A lot of this week's talks will use it.

Ingredients

Fluid equations

Continuity equation (mass conservation)

Mass conservation: Classically, matter is neither created nor destroyed.

Define the mass density as

$$\rho(\mathbf{x}, t) = \lim_{\delta V \to 0} \frac{\delta M}{\delta V} \qquad M(V, t) = \iiint_V \rho(\mathbf{x}, t)\, \mathrm{d}V$$

Using the 3D Leibnitz theorem with boundary moving with the fluid

$$\frac{\mathrm{d}M}{\mathrm{d}t} = \iiint_{V(t)} \left(\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{u}) \right) \mathrm{d}V = 0$$

by conservation of mass

If true for any volume of fluid, then everywhere

$$\boxed{\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{u}) = 0}$$

Convective derivative

The total time derivative when moving with the fluid is called the *convective derivative*. Using the chain rule it is given by

$$\frac{\mathrm{D}}{\mathrm{Dt}} \equiv \frac{\partial}{\partial t} + \mathbf{u} \cdot \nabla$$

Capital $\mathbf{D}$ used to signify special choice for velocity

Contribution from local changes in time

Contribution from spatial derivatives

Momentum equation (Newton's 2nd law)

Newton's 2nd law: Sum of the forces on object equals its rate of change of momentum ($\mathbf{F} = m\mathbf{a}$). For a fluid:

$$\text{sum of forces on } V = \frac{\mathrm{d}}{\mathrm{d}t}\iiint_V \rho\mathbf{u}\,\mathrm{d}V = \iiint_V \rho\frac{\mathrm{D}\mathbf{u}}{\mathrm{D}t}\,\mathrm{d}V$$

by 3D Leibniz theorem
and mass continuity

Consider two types of forces…

Momentum equation (Newton's 2nd law)

Body forces

act within the volume

Introduce $\mathbf{F}_b$ as the body force per unit volume.

$$\iiint_V \mathbf{F}_b \, \mathrm{d}V$$

For example, gravity adds

$$\mathbf{F}_g = \lim_{\delta V \to 0} \left\{ \frac{\mathbf{g} \delta M}{\delta V} \right\} = \rho \mathbf{g}$$

Contact forces

act on the boundary

If the only contact force is pressure acting normal to the surface, $-p\hat{\mathbf{n}}$, the total contact force on V is

$$\iint_{\partial V} -p\hat{\mathbf{n}} \, \mathrm{d}S = \iiint_V -\nabla p \, \mathrm{d}V$$

Note: Using a pressure gradient for contact forces is dubious, but we'll save that discussion for later.

Momentum equation (Newton's 2nd law)

Matching the two sides of Newton's 2nd law therefore gives

$$\iiint_V \rho \frac{\mathrm{D}\mathbf{u}}{\mathrm{D}t}\,\mathrm{d}V = \iiint_V \left(\mathbf{F}_b - \nabla p\right)\,\mathrm{d}V$$

If true for any volume of fluid, then everywhere

$$\boxed{\rho \frac{\mathrm{D}\mathbf{u}}{\mathrm{D}t} = \mathbf{F}_b - \nabla p}$$

Summary of fluid equations

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{u}) = 0$$

$$\rho \frac{\mathrm{D}\mathbf{u}}{\mathrm{D}t} = \mathbf{F}_b - \nabla p$$

$$\frac{\mathrm{D}p}{\mathrm{D}t} = -\gamma p \nabla \cdot \mathbf{u} + (\gamma - 1) Q$$

From first law of thermodynamics and LTE closure

$$\epsilon = (\gamma - 1)\frac{p}{\rho} \qquad \gamma = \frac{s+2}{s}$$

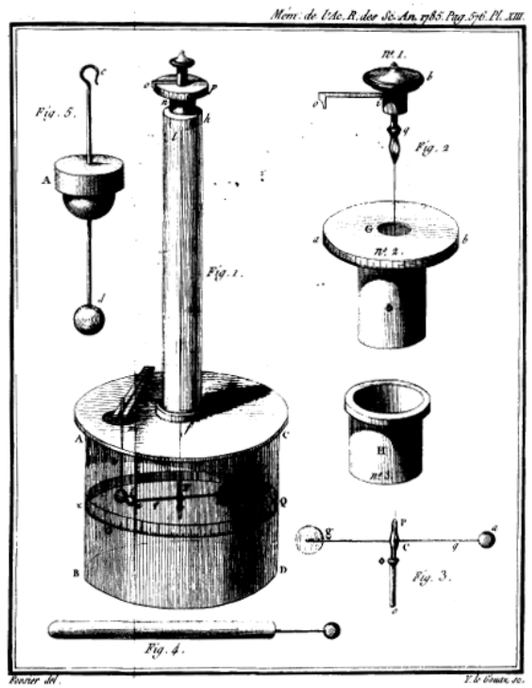
Electromagnetism

Maxwell's Equations

	$\mathbf{E}$	$\mathbf{B}$
Div	$\nabla \cdot \mathbf{E} = \frac{\rho_c}{\epsilon_0}$	$\nabla \cdot \mathbf{B} = 0$
Curl	$\nabla \times \mathbf{E} + \frac{\partial \mathbf{B}}{\partial t} = 0$	$\nabla \times \mathbf{B} - \frac{1}{c^2}\frac{\partial \mathbf{E}}{\partial t} = \mu_0 \mathbf{j}$

The homogenous equations allow us to use potentials, reducing the number of variables. The resulting equations look especially great in 4D space-time. However, deriving and understanding MHD goes better if we use the equations above, which are closer to applications.

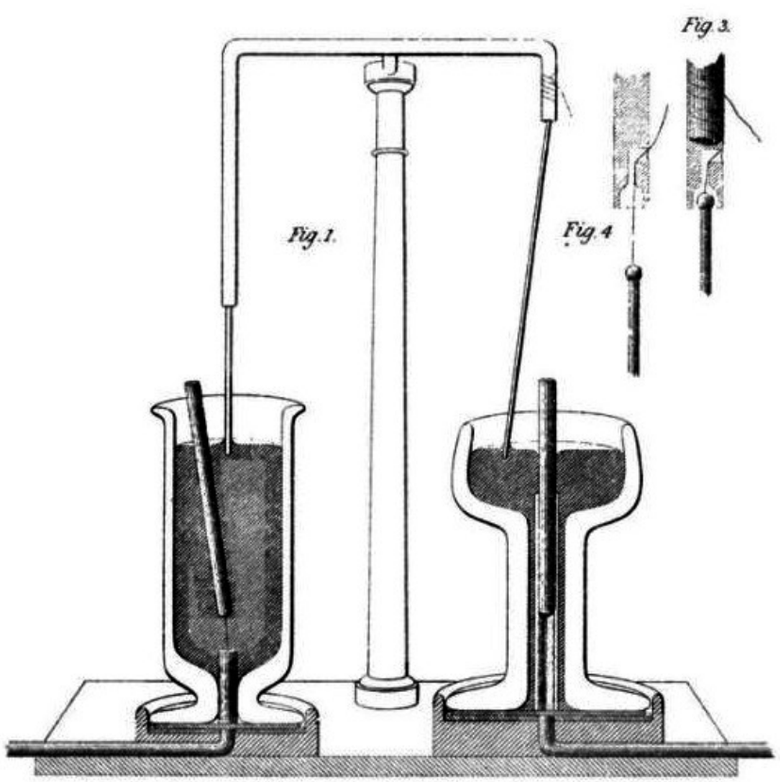
Electromagnetic forces (motors)



Electrostatic force per unit volume

$$\mathbf{F}_C = \rho_c \mathbf{E}$$

(from Coulomb's experiments, 1784)



Lorentz force per unit volume from a current

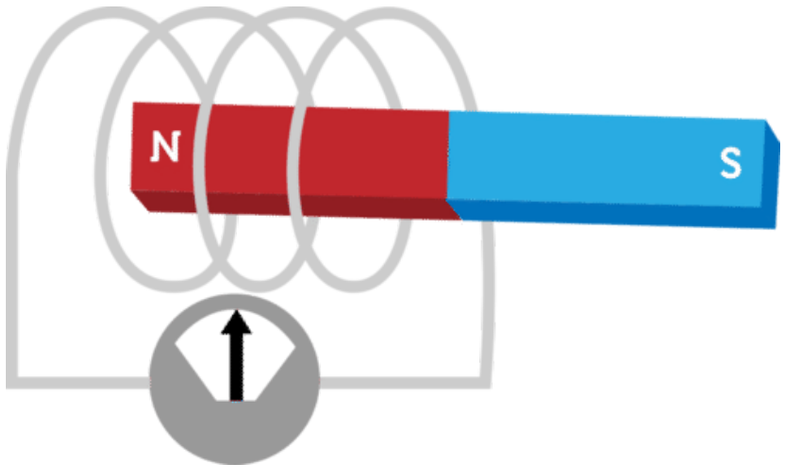
$$\mathbf{F}_L = \mathbf{j} \times \mathbf{B}$$

(from Ampère's and Faraday's experiments, 1821-23)

Ohm's law for conductors (generators)

For many materials on a lab bench, observe $\mathbf{j} = \sigma \mathbf{E}$

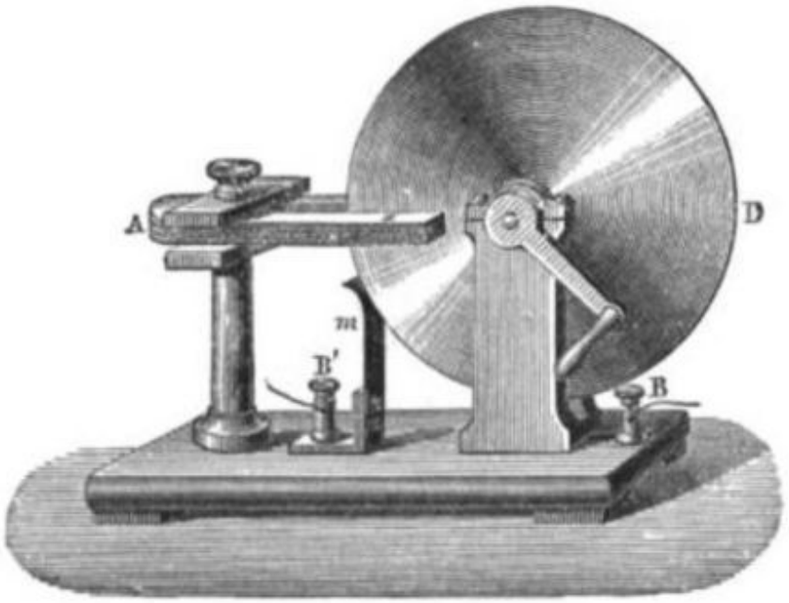
Induction:



$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}$$

The more general relation that fixes the paradox is

$$\mathbf{j} = \sigma \left(\mathbf{E} + \mathbf{v} \times \mathbf{B} \right)$$



Electromagnetic energy and Poynting's theorem (1884)

$$\frac{\partial U}{\partial t} + \nabla \cdot \mathbf{S} = -\mathbf{E} \cdot \mathbf{j} \qquad U = \frac{\epsilon_0 E^2}{2} + \frac{B^2}{2\mu_0} \qquad \mathbf{S} = \frac{1}{\mu_0} \mathbf{E} \times \mathbf{B}$$

How is energy converted with Ohm's law?

$$\mathbf{j} = \sigma \left(\mathbf{E} + \mathbf{v} \times \mathbf{B} \right) \quad \Leftrightarrow \quad \mathbf{E} = -\mathbf{v} \times \mathbf{B} + \frac{\mathbf{j}}{\sigma}$$

work done on conductor

$$-\mathbf{E} \cdot \mathbf{j} = \mathbf{v} \times \mathbf{B} \cdot \mathbf{j} - \frac{j^2}{\sigma} = \mathbf{v} \cdot \mathbf{B} \times \mathbf{j} - \frac{j^2}{\sigma} = -\mathbf{v} \cdot \mathbf{j} \times \mathbf{B} - \frac{j^2}{\sigma}$$

resistive heating

Summary of key E.M. results

Forces

$$\mathbf{F}_C = \rho_c \mathbf{E}$$

$$\mathbf{F}_L = \mathbf{j} \times \mathbf{B}$$

Maxwell's equations

$$\nabla \cdot \mathbf{E} = \frac{\rho_c}{\epsilon_0} \qquad \nabla \times \mathbf{E} + \frac{\partial \mathbf{B}}{\partial t} = 0$$

$$\nabla \cdot \mathbf{B} = 0 \qquad \nabla \times \mathbf{B} - \frac{1}{c^2}\frac{\partial \mathbf{E}}{\partial t} = \mu_0 \mathbf{j}$$

Poynting's theorem

$$U = \frac{\epsilon_0 E^2}{2} + \frac{B^2}{2\mu_0} \qquad \mathbf{S} = \frac{1}{\mu_0}\mathbf{E} \times \mathbf{B} \qquad \frac{\partial U}{\partial t} + \nabla \cdot \mathbf{S} = -\mathbf{E} \cdot \mathbf{j}$$

Ohm's law for a conductor

$$\mathbf{E} + \mathbf{v} \times \mathbf{B} = \frac{\mathbf{j}}{\sigma}$$

Heating

$$Q_{\text{ohmic}} = \frac{j^2}{\sigma}$$

Derivation of resistive MHD

Fluid equations

$$\frac{\partial \rho}{\partial t} + \nabla \cdot \left(\rho \mathbf{u}\right) = 0$$

$$\rho \frac{\mathrm{D}\mathbf{u}}{\mathrm{D}t} = \mathbf{F}_b - \nabla p$$

$$\frac{\mathrm{D}p}{\mathrm{D}t} = -\gamma p \nabla \cdot \mathbf{u} + \left(\gamma - 1\right) Q$$

E.M. equations

$$\nabla \cdot \mathbf{E} = \frac{\rho_c}{\epsilon_0} \qquad \nabla \times \mathbf{E} + \frac{\partial \mathbf{B}}{\partial t} = 0$$

$$\nabla \cdot \mathbf{B} = 0 \qquad \nabla \times \mathbf{B} - \frac{1}{c^2}\frac{\partial \mathbf{E}}{\partial t} = \mu_0 \mathbf{j}$$

$$\mathbf{F}_L = \rho_c \mathbf{E} + \mathbf{j} \times \mathbf{B} \qquad \mathbf{E} + \mathbf{u} \times \mathbf{B} = \frac{\mathbf{j}}{\sigma} \qquad Q_{Ohmic} = \frac{j^2}{\sigma}$$

The MHD Approximation

Standard MHD assumes that speeds are much less than c.

Order of Magnitude Analysis

Let l_0 be a typical length scale, t_0 a typical time scale, $v_0 \sim l_0/t_0$, and let E_0 and B_0 be typical values of $\left|\mathbf{E}\right|$ and $\left|\mathbf{B}\right|$.

$$\nabla \times \mathbf{E} + \frac{\partial \mathbf{B}}{\partial t} = 0 \;\Rightarrow\; \frac{E_0}{l_0} \sim \frac{B_0}{t_0} \;\Rightarrow\; E_0 \sim v_0 B_0 \,.$$

Now compare terms on LHS of $\nabla \times \mathbf{B} - \cancel{\frac{1}{c^2}\frac{\partial \mathbf{E}}{\partial t}} = \mu_0 \mathbf{j}$.

$$\frac{\left|\frac{1}{c^2}\frac{\partial \mathbf{E}}{\partial t}\right|}{|\nabla \times \mathbf{B}|} \sim \frac{E_0/(t_0 c^2)}{B_0/l_0} \sim \left(\frac{v_0}{c}\right)^2 \ll 1.$$

MHD Ampère's law is $\mathbf{j} = \dfrac{1}{\mu_0}\nabla \times \mathbf{B}$ (aka. low-frequency Ampère's law).

Fluid equations

$$\frac{\partial \rho}{\partial t} + \nabla \cdot \left(\rho \mathbf{u}\right) = 0$$

$$\rho \frac{\mathrm{D}\mathbf{u}}{\mathrm{D}t} = \mathbf{F}_b - \nabla p$$

$$\frac{\mathrm{D}p}{\mathrm{D}t} = -\gamma p \nabla \cdot \mathbf{u} + \left(\gamma - 1\right) Q$$

E.M. equations

$$\nabla \cdot \mathbf{E} = \frac{\rho_c}{\epsilon_0} \qquad \nabla \times \mathbf{E} + \frac{\partial \mathbf{B}}{\partial t} = 0$$

$$\nabla \cdot \mathbf{B} = 0 \qquad \nabla \times \mathbf{B} - \cancel{\frac{1}{c^2}\frac{\partial \mathbf{E}}{\partial t}} = \mu_0 \mathbf{j}$$

$$\mathbf{F}_L = \cancel{\rho_c \mathbf{E}} + \mathbf{j} \times \mathbf{B} \qquad \mathbf{E} + \mathbf{u} \times \mathbf{B} = \frac{\mathbf{j}}{\sigma} \qquad Q_{Ohmic} = \frac{j^2}{\sigma}$$

Fluid equations

$$\frac{\partial \rho}{\partial t} + \nabla \cdot \left(\rho \mathbf{u}\right) = 0$$

$$\rho \frac{\mathrm{D}\mathbf{u}}{\mathrm{D}t} = \mathbf{F}_b - \nabla p$$

$$\frac{\mathrm{D}p}{\mathrm{D}t} = -\gamma p \nabla \cdot \mathbf{u} + \left(\gamma - 1\right) Q$$

E.M. equations

$$\nabla \cdot \mathbf{E} = \frac{\rho_c}{\epsilon_0} \qquad \nabla \times \mathbf{E} + \frac{\partial \mathbf{B}}{\partial t} = 0$$

$$\nabla \cdot \mathbf{B} = 0 \qquad \mathbf{j} = \frac{1}{\mu_0} \nabla \times \mathbf{B}$$

$$\mathbf{F}_L = \mathbf{j} \times \mathbf{B} \qquad \mathbf{E} + \mathbf{u} \times \mathbf{B} = \frac{\mathbf{j}}{\sigma} \qquad Q_{Ohmic} = \frac{j^2}{\sigma}$$

Fluid equations

$$\frac{\partial \rho}{\partial t} + \nabla \cdot \left(\rho \mathbf{u}\right) = 0$$

$$\rho \frac{\mathrm{D}\mathbf{u}}{\mathrm{D}t} = \mathbf{j} \times \mathbf{B} + \rho \mathbf{g} - \nabla p$$

$$\frac{\mathrm{D}p}{\mathrm{D}t} = -\gamma p \nabla \cdot \mathbf{u} + \left(\gamma - 1\right) \frac{j^2}{\sigma}$$

E.M. equations

$$\left(\nabla \cdot \mathbf{E} = \frac{\rho_c}{\epsilon_0}\right)$$

$$\nabla \times \mathbf{E} + \frac{\partial \mathbf{B}}{\partial t} = 0$$

$$\left(\nabla \cdot \mathbf{B} = 0\right)$$

$$\mathbf{j} = \frac{1}{\mu_0} \nabla \times \mathbf{B}$$

$$\mathbf{E} + \mathbf{u} \times \mathbf{B} = \frac{\mathbf{j}}{\sigma}$$

MHD equations

Mass continuity equation: $$\frac{\partial \rho}{\partial t} + \nabla \cdot \left(\rho \mathbf{u}\right) = 0$$

$$\mathbf{j} = \frac{1}{\mu_0} \nabla \times \mathbf{B}$$

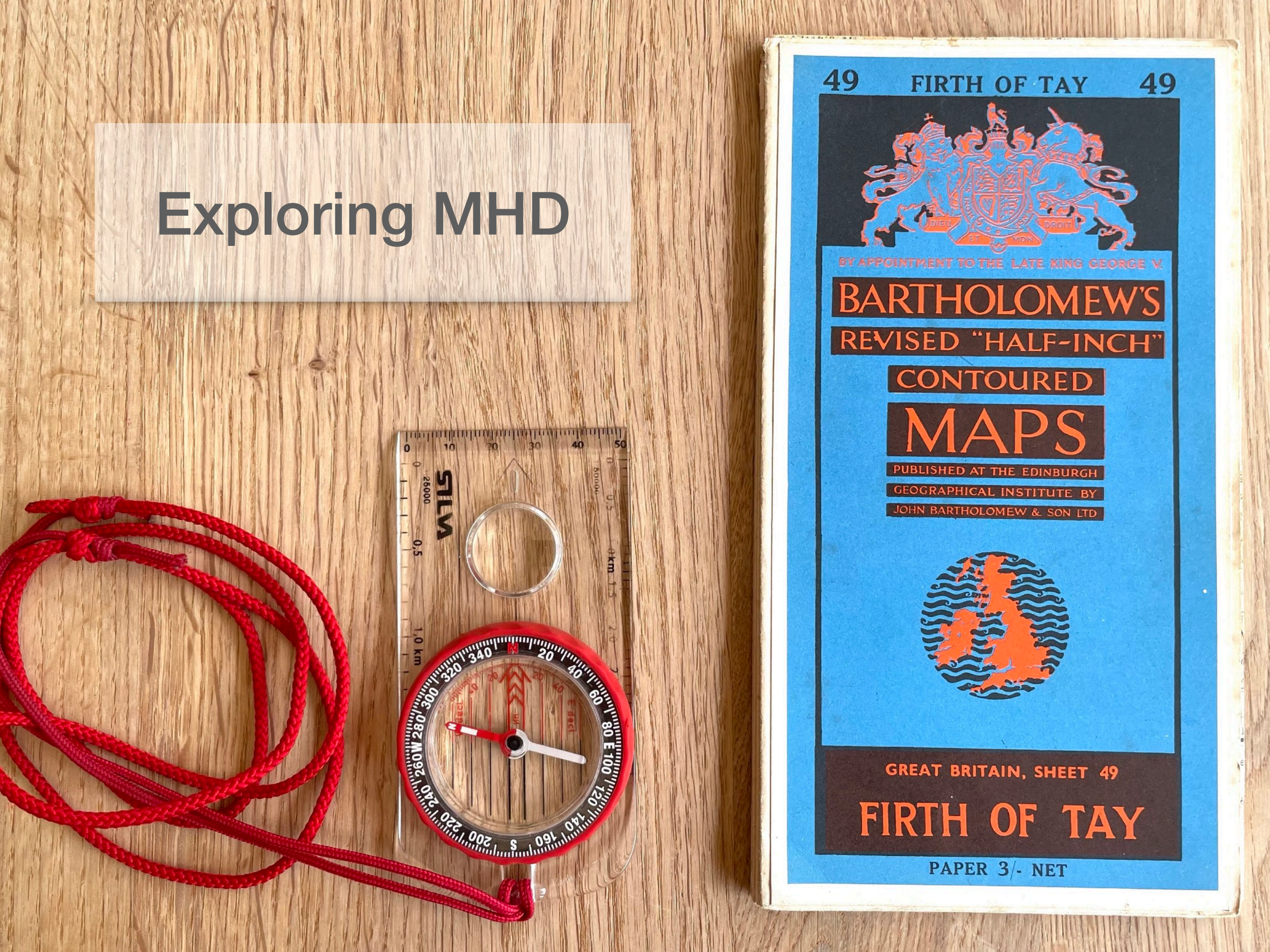
Momentum equation: $$\rho \frac{\mathrm{D}\mathbf{u}}{\mathrm{D}t} = \mathbf{j} \times \mathbf{B} + \rho \mathbf{g} - \nabla p$$

Energy equation: $$\frac{\mathrm{D}p}{\mathrm{D}t} = -\gamma p \nabla \cdot \mathbf{u} + \left(\gamma - 1\right) \frac{j^2}{\sigma}$$

$$\eta = \frac{1}{\mu_0 \sigma}$$

Induction equation: $$\frac{\partial \mathbf{B}}{\partial t} = \nabla \times (\mathbf{u} \times \mathbf{B}) - \nabla \times \left(\eta \nabla \times \mathbf{B}\right)$$

Exploring MHD



Induction Equation and R_m

$$\eta = \frac{1}{\mu_0 \sigma}$$

The general resistive induction equation is:

$$\frac{\partial \mathbf{B}}{\partial t} = \nabla \times (\mathbf{u} \times \mathbf{B}) - \nabla \times \left(\eta \nabla \times \mathbf{B}\right).$$

If the conductivity is constant in space (good luck making that assumption in reality!), we equivalently have:

$$\frac{\partial \mathbf{B}}{\partial t} = \underbrace{\nabla \times (\mathbf{u} \times \mathbf{B})}_{\text{advection}} + \underbrace{\cancel{\eta \nabla^2 \mathbf{B}}}_{\text{diffusion}}.$$

Comparing size of terms: $$\frac{\left|\nabla \times (\mathbf{v} \times \mathbf{B})\right|}{\left|\eta \nabla^2 \mathbf{B}\right|} \sim \frac{u_0 B_0 / l_0}{\eta B_0 / l_0^2} \sim \frac{l_0 u_0}{\eta} =: R_m .$$

R_m is called the *magnetic Reynolds number*.

Often, R_m is huge, e.g. 10^8–10^{12} common in the solar corona.

Ideal MHD

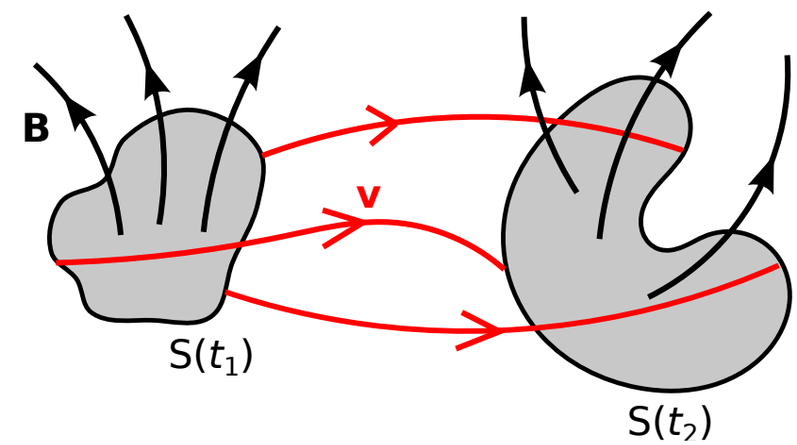
In a highly conducting fluid, the electric field is zero in the frame of the fluid.

When $R_m \gg 1$, use ideal induction equation $\dfrac{\partial \mathbf{B}}{\partial t} = \nabla \times (\mathbf{u} \times \mathbf{B})$.

Alfvén's Theorem

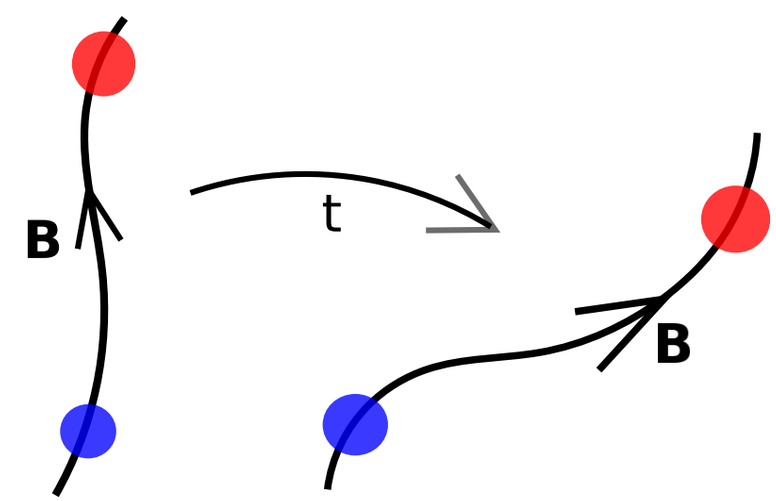
In ideal MHD, the magnetic flux through any surface co-moving with $\mathbf{u}$ is conserved:

$$\frac{\mathrm{d}}{\mathrm{d}t} \iint_{S(t)} \mathbf{B} \cdot \mathrm{d}\mathbf{S} = 0$$



Field Line Conservation

In ideal MHD, two fluid elements lying on the same magnetic field line at some initial time always do so



Magnetic Forces

Momentum equation: $$\rho\left(\frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u}\cdot\nabla)\mathbf{u}\right) = \mathbf{j}\times\mathbf{B} + \rho\mathbf{g} - \nabla p$$

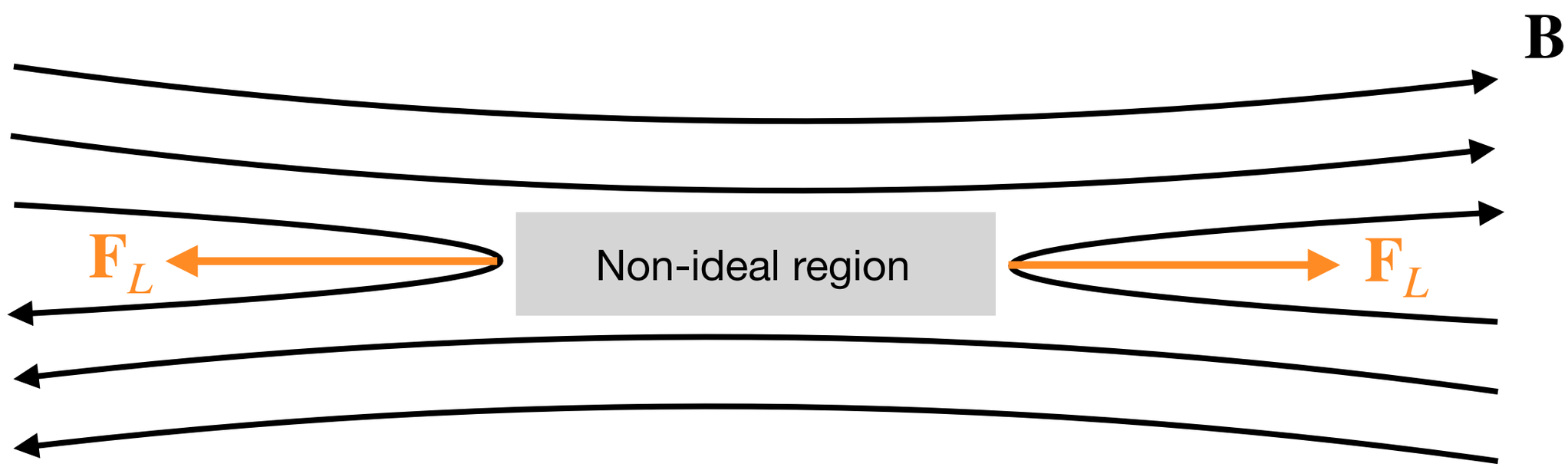
Lorentz force term can be expanded as:

$$\mathbf{j}\times\mathbf{B} = \frac{1}{\mu_0}(\nabla\times\mathbf{B})\times\mathbf{B} = \underbrace{-\nabla\left(\frac{B^2}{2\mu_0}\right)}_{\text{magnetic pressure}} + \underbrace{\frac{1}{\mu_0}(\mathbf{B}\cdot\nabla)\mathbf{B}}_{\text{magnetic tension}}.$$

Alternative decomposition, terms perpendicular to $\mathbf{B}$:

$$\mathbf{j}\times\mathbf{B} = \frac{1}{\mu_0}(\nabla\times\mathbf{B})\times\mathbf{B} = \underbrace{-\nabla_\perp\left(\frac{B^2}{2\mu_0}\right)}_{\text{magnetic pressure}} + \underbrace{\frac{B^2}{\mu_0}(\hat{\mathbf{b}}\cdot\nabla)\hat{\mathbf{b}}}_{\text{magnetic tension}}.$$

Example: Magnetic Reconnection



Where is plasma being accelerated?

What's special about the non-ideal region?

Force free fields

Momentum equation: $$\rho\left(\frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u}\cdot\nabla)\mathbf{u}\right) = \mathbf{j}\times\mathbf{B} + \rho\mathbf{g} - \nabla p$$

Define the *plasma beta* as the pressure ratio $\beta := \dfrac{p}{B^2/2\mu_0}$.

Pressure force $\ll$ magnetic forces when $\beta \ll 1$ (cold plasma).

Gravity also often negligible, hence seek static equilibria with:

$$\mathbf{j}\times\mathbf{B} = \mathbf{0}\,.$$

Force-free magnetic fields have force balance between magnetic tension and magnetic pressure.

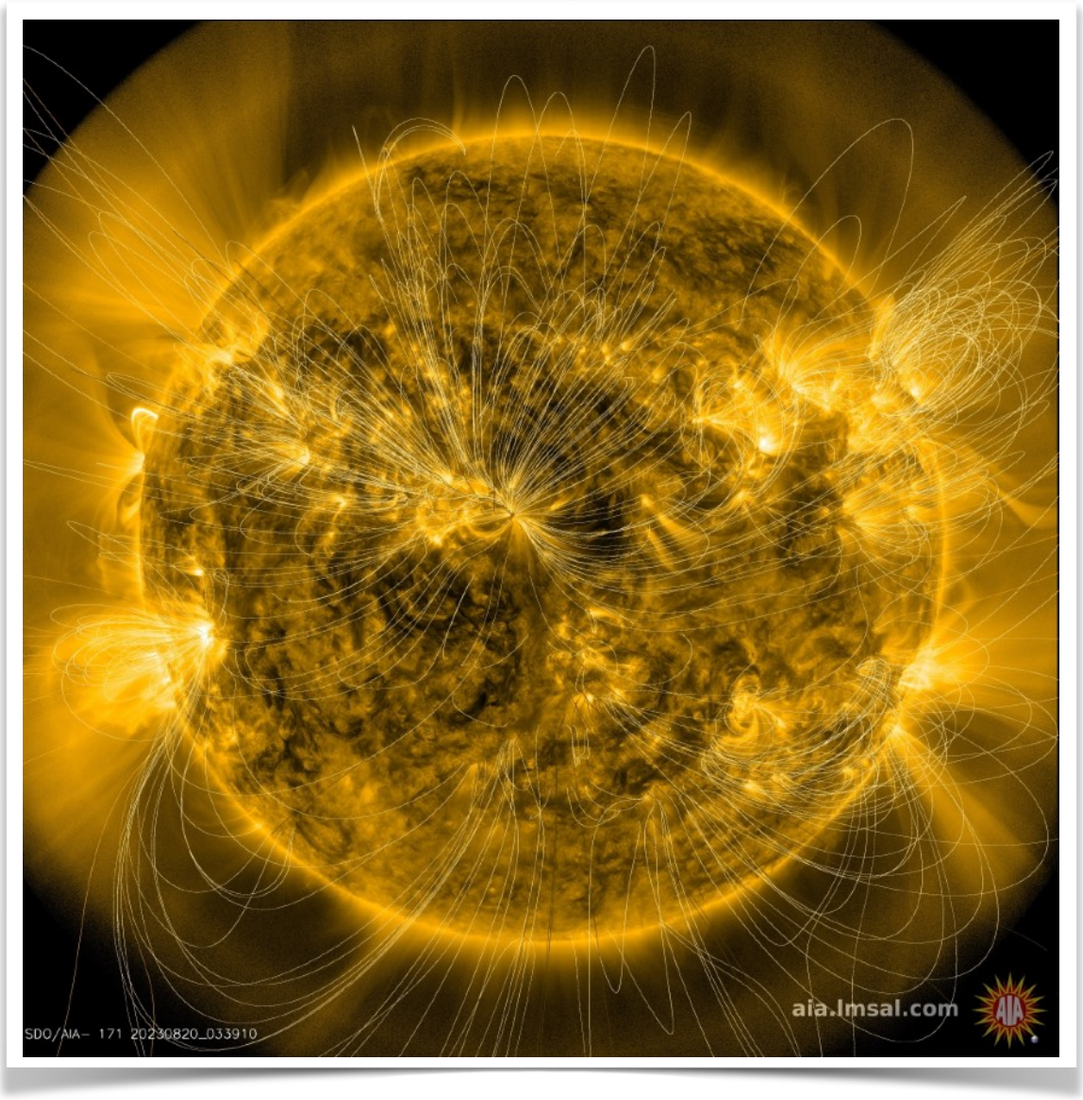
$$(\nabla \times \mathbf{B}) \times \mathbf{B} = \mathbf{0} \;\; \Leftrightarrow \;\; \nabla \times \mathbf{B} = \alpha(\mathbf{x})\mathbf{B}$$

Taking div leads to $\mathbf{B} \cdot \nabla \alpha = 0$, i.e. α constant along field lines.

Special cases

Potential field, $\nabla \times \mathbf{B} = \mathbf{0}$, lowest magnetic energy given BCs.

Linear force free field, $\alpha =$ const, lowest magnetic energy given BCs *and* magnetic helicity.



Warning: There Be Dragons

The macroscopic derivation shown in this talk
is generally sound, however…

Formal derivation takes moments of the Boltzmann equation to
obtain multi-fluid equations, then combines into a single fluid model.

Broader Horizons 1: Generalised Ohm's law

$$\mathbf{E} + \mathbf{u} \times \mathbf{B} = \frac{\mathbf{j}}{\sigma} + \frac{1}{ne}\left(\mathbf{j} \times \mathbf{B} - \nabla \cdot \mathbf{P}_e\right) + \frac{m_e}{ne^2}\left(\frac{\partial \mathbf{j}}{\partial t} + \nabla \cdot \left(\mathbf{uj} + \mathbf{ju} - \frac{\mathbf{jj}}{ne}\right)\right).$$

Ideal MHD remains a good approximation at large scales,
but Hall (and electron pressure) effects often at least as important
as resistivity where ideal MHD breaks down.

Neutrals can also affect Ohm's law, e.g. Cowling resistivity.

Broader Horizons 2: Internal Forces

$$\rho\left(\frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u}\cdot\nabla)\mathbf{u}\right) = \mathbf{j}\times\mathbf{B} + \rho\mathbf{g} - \nabla\cdot\mathbf{P}$$

Gryomotion of particles around $\mathbf{B}$ produces gyrotropic pressure

$$\mathbf{P} = p_{\perp}\mathbf{I} + (p_{\parallel} - p_{\perp})\hat{\mathbf{b}}\hat{\mathbf{b}}$$

$-\nabla\cdot\mathbf{P}$ reduces to $-\nabla p$ when collisions make $p_{\parallel} \approx p_{\perp}$.

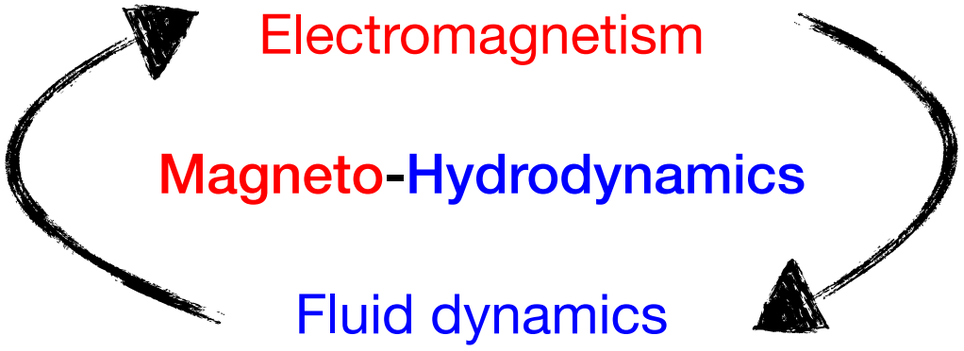
When $\lambda_{mfp} \gtrsim l_0$ (e.g. outer corona, solar wind…) pressure isotropy is a bad approximation. More suitable fluid approaches include:

Braginskii MHD, CGL equations and Landau MHD.

Additionally, some models explicitly treat higher-order moments, or include gyroradius effects, or have dust, or are relativistic…

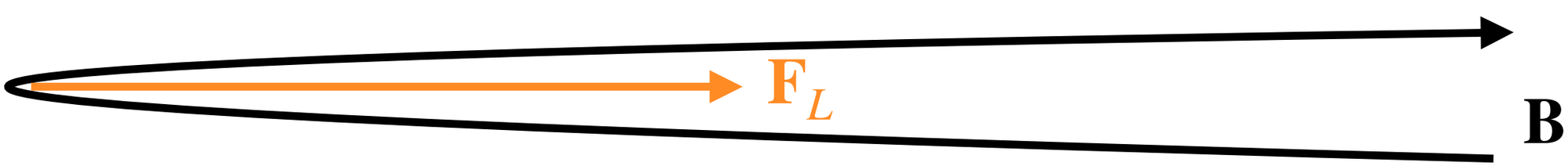
Summary

Magnetohydrodynamics (MHD) describes dynamics of electrically conducting fluid (e.g. plasma) coupled to magnetic field.



Ideal MHD when $R_m := \dfrac{l_0 u_0}{\eta} \gg 1$ (“normal”). Field lines “frozen in”.

Magnetic forces can be expressed as a sum of magnetic pressure and magnetic tension forces, e.g. $\mathbf{j} \times \mathbf{B} = -\nabla\left(\dfrac{B^2}{2\mu_0}\right) + \dfrac{1}{\mu_0}(\mathbf{B}\cdot\nabla)\mathbf{B}$.



Magnetic fields are often approximated as force-free, $\mathbf{j} \times \mathbf{B} = \mathbf{0}$.

MHD equations

Mass continuity equation: $$\frac{\partial \rho}{\partial t} + \nabla \cdot \left(\rho \mathbf{u}\right) = 0$$

$$\mathbf{j} = \frac{1}{\mu_0} \nabla \times \mathbf{B}$$

Momentum equation: $$\rho \frac{\mathrm{D}\mathbf{u}}{\mathrm{D}t} = \mathbf{j} \times \mathbf{B} + \rho \mathbf{g} - \nabla p$$

Energy equation: $$\frac{\mathrm{D}p}{\mathrm{D}t} = -\gamma p \nabla \cdot \mathbf{u} + \left(\gamma - 1\right) \frac{j^2}{\sigma}$$

$$\eta = \frac{1}{\mu_0 \sigma}$$

Induction equation: $$\frac{\partial \mathbf{B}}{\partial t} = \nabla \times (\mathbf{u} \times \mathbf{B}) - \nabla \times \left(\eta \nabla \times \mathbf{B}\right)$$

Modelling plasma in a fluid approach can get more complicated than this, but standard MHD is simple enough to gain valuable insights, and rich enough to keep you interested for a lifetime!